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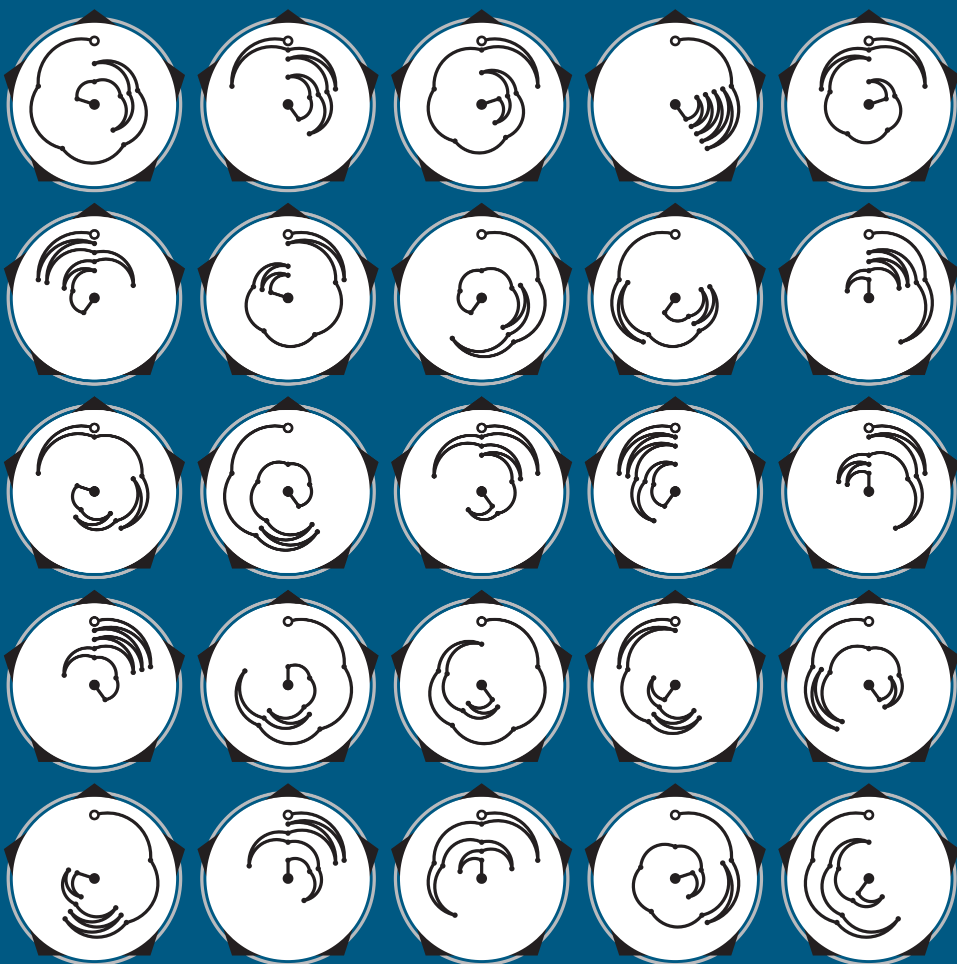
MAA

MATHEMATICS MAGAZINE



- Nested radicals, Jenny, and pi
- Fibonacci, Lucas, and the dice game *LCR*
- Recurrences and powers of generating functions
- A set of analysis articles and a review of *The Joy of SET*

EDITORIAL POLICY

Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the Magazine. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

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COVER IMAGE

LCR-5-10 © 2016 David A. Reimann (*Albion College*). Used by permission.

The paper by Torrence and Torrence was the inspiration for the cover art. Illustrated in a five by five grid are graphical representations of twenty-five of the possible endgames of LCR with five players, each ending with ten non-center moves. The lines represent the movement of the final chip from the original player to the center.

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LETTER FROM THE EDITOR

This issue starts off with an article about the convergence of periodic left-nested radicals by Devyn Lesher and Chris Lynd. Besides theoretical results, they also construct a sequence of left-nested radicals that asymptotically converges to a periodic sequence that repeats the digits of Jenny's phone number: 867-5309. Following the nested radical theme, Mu-Ling Chang and Chia-Chin Chang offer a note in which they use nested radicals to evaluate π.

My wife's family plays the popular dice game *Left, Center, Right* or *LCR* every year at the winter holidays. Bruce Torrence and Robert Torrence consider a variation of this game, connecting the endgame to Fibonacci and Lucas numbers. This article was the inspiration for Dave Reimann's cover art for this issue. Finbarr Holland provides a characterization of quadratic polynomials, motivated by an article from the MAA's *Focus*.

Interspersed in this issue are two proofs without words. In one, Tom Edgar looks at factorial sums. In the other, Ángel Plaza considers alternating sums of binomial coefficients in Pascal's triangle.

Fibonacci and Lucas make another appearance in the next article. Chebyshev, Fibonacci, Lucas, Pell, Jacobsthal, Morgan-Voyce, and Fermat polynomial sequences have polynomial rational functions of the same general form as their generating functions. Raymond Beauregard and Vladimir Dobrushkin provide a recurrence for the general rational function, allowing a speedy way to compute terms in the associated sequences.

The next three articles form a set of analysis articles. Russell Gordon demonstrates a bounded derivative that is not Riemann integrable. George Stoica generates a continuous, nowhere-differentiable function, and Mark Lynch considers a function with a continuous, nonzero derivative whose inverse is nowhere continuous.

This issue includes an interview with Anne Burns, a retired mathematics professor who enjoys creating digital art influenced by mathematics and the natural world. There is also a crossword puzzle by Brendan Sullivan in advance of the 2017 Joint Mathematics Meetings. The regular departments of Problems and Reviews follow. The issue concludes with the problems and solutions from, as well as details about, the 57th International Mathematical Olympiad, as well as a thank you to referees.

Michael A. Jones, Editor

ARTICLES

Convergence Results for the Class of Periodic Left Nested Radicals

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The equation

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}=2$$

states that the corresponding sequence of finite sums

$$1,\ \ 1+\frac{1}{2},\ \ 1+\frac{1}{2}+\frac{1}{4},\ \ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8},\ \ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16},\ \ \ldots$$

has a limit of two.

Similarly, the equation involving the *right* nested radical

$$\sqrt{6+\sqrt[3]{-7-\sqrt[4]{3-\sqrt{6+\sqrt[3]{-7-\sqrt[4]{3-\cdots}}}}}}=2$$

states that the corresponding sequence of finite nested radicals

$$\sqrt{6},\ \sqrt{6+\sqrt[3]{-7}},\ \sqrt{6+\sqrt[3]{-7-\sqrt[4]{3}}},\ \sqrt{6+\sqrt[3]{-7-\sqrt[4]{3-\sqrt{6}}}},\ \ldots$$

has a limit of two. Also, the equation involving the *left* nested radical

$$\cdots-\sqrt[4]{14+\sqrt[3]{10-\sqrt{6-\sqrt[4]{14+\sqrt[3]{10-\sqrt{6}}}}}}=2$$

states that the corresponding sequence of finite nested radicals

$$\sqrt{6},\ \sqrt[3]{10-\sqrt{6}},\ \sqrt[4]{14+\sqrt[3]{10-\sqrt{6}}},\ \sqrt{6-\sqrt[4]{14+\sqrt[3]{10-\sqrt{6}}}},\ \ldots$$

has a limit of two.

Math. Mag. **89** (2016) 319–335. doi:10.4169/math.mag.89.5.319. © Mathematical Association of America
MSC: Primary 39A23; Secondary 40A05

What is $\cdots - \sqrt{22 + \sqrt[3]{25 + \sqrt[4]{21 - \sqrt{22 + \sqrt[3]{25 + \sqrt[4]{21}}}}}}$ **?**

The corresponding sequence of left nested radicals is

$$\sqrt[4]{21},\ \sqrt[3]{25+\sqrt[4]{21}},\ \sqrt{22+\sqrt[3]{25+\sqrt[4]{21}}},\ \sqrt[4]{21-\sqrt{22+\sqrt[3]{25+\sqrt[4]{21}}}},\ \ldots$$

and the sequence does not have a limit. Instead, it asymptotically converges to a periodic sequence that repeats the three numbers 2, 3, 5.

Is there a nested radical whose limiting sequence repeats the digits 867-5309?

Tommy Tutone fans will be happy to know that we can construct a left nested radical whose computed sequence becomes an endless repetition of Jenny's phone number. However, our nested radical's limit is *not* a period-7 sequence! See Example 7.

There are two fundamental questions that are found throughout the research on nested radicals: (1) Given a particular form of nested radical, under what conditions does it converge? (2) Given a particular form of nested radical, which numbers can be expressed as the limit of a nested radical? The papers [**2**]–[**7**] address one or both of these research questions.

In this paper we address both research questions, as they pertain to periodic *left* nested radicals. We provide numerical examples to illustrate each result.

We also provide a recipe for constructing nested radicals with a predetermined end-behavior. If you choose the form of the nested radical and a limit, the recipe shows how to construct the *unique* nested radical of the chosen form that converges to the chosen limit. If you choose a periodic limiting sequence, the recipe shows how to construct a nested radical that asymptotically converges to the chosen periodic sequence.

Background

In this section we provide the definition of a left and a right nested radical. We also present two theorems to help illustrate some of the questions that are open for exploration in the area of left nested radicals. For a brief summary of the research on nested radicals from 1911–2012, see [**4**].

The following definition provides the general form for left and right nested radicals.

Definition 1. Let $\{r_n\}$ be a sequence of integers that are greater than or equal to two. Let $\{c_n\}$ and $\{a_n\}$ be sequences of real numbers. If

$$\{z_n\} = \sqrt[r_1]{a_1},\ \sqrt[r_1]{a_1 + c_1\sqrt[r_2]{a_2}},\ \ldots,\ \sqrt[r_1]{a_1 + c_1\sqrt[r_2]{a_2 + \cdots + c_{n-1}\sqrt[r_n]{a_n}}},\ \ldots$$

is a sequence of real numbers, then the expression

$$\sqrt[r_1]{a_1 + c_1\sqrt[r_2]{a_2 + c_2\sqrt[r_3]{a_3 + c_3\sqrt[r_4]{a_4 + \cdots}}}}$$

is called a right nested radical and it denotes the limit of $\{z_n\}$, if it exists.

Similarly, if

$$\{y_n\} = \sqrt[r_1]{a_1},\ \sqrt[r_2]{a_2 + c_1\sqrt[r_1]{a_1}},\ \ldots,\ \sqrt[r_n]{a_n + \cdots + c_2\sqrt[r_2]{a_2 + c_1\sqrt[r_1]{a_1}}},\ \ldots$$

is a sequence of real numbers, then the expression

$$\cdots + c_4 \sqrt[r_4]{a_4 + c_3 \sqrt[r_3]{a_3 + c_2 \sqrt[r_2]{a_2 + c_1 \sqrt[r_1]{a_1}}}}$$

is called a left nested radical and it denotes the limit of $\{y_n\}$, if it exists.

The sequence $\{r_n\}$ is referred to as the corresponding sequence of indices, $\{c_n\}$ is the corresponding sequence of coefficients, and $\{a_n\}$ is the corresponding sequence of radicands.

The next example shows how to combine the sequences $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$ to make a sequence of nested radicals. As an online supplement to this paper, we provide the numerical calculations for each example. See [**8**] for the URL.

Example 1. Let the sequence of indices be the periodic sequence $\{r_n\} = 3, 2, 3, 2, \ldots$, let the sequence of coefficients be the periodic sequence $\{c_n\} = -1, 1, -1, 1, \ldots$, and let the sequence of radicands be the constant sequence $\{a_n\} = 6, 6, 6, \ldots$.

By Definition 1, the sequence of *left* nested radicals is

$$\{y_n\} = \sqrt[3]{6},\ \sqrt{6 + (-1)\sqrt[3]{6}},\ \sqrt[3]{6 + \sqrt{6 + (-1)\sqrt[3]{6}}},$$
$$\sqrt{6 + (-1)\sqrt[3]{6 + \sqrt{6 + (-1)\sqrt[3]{6}}}},\ \ldots,$$

which converges to two. Thus, we have the equation

$$\cdots + (-1)\sqrt[3]{6 + \sqrt{6 + (-1)\sqrt[3]{6 + \sqrt{6 + (-1)\sqrt[3]{6}}}}} = 2.$$

Each negative coefficient can be rewritten using a subtraction sign, so we have

$$\cdots - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6 + \sqrt{6 - \sqrt[3]{6}}}}} = 2.$$

By Definition 1, the sequence of *right* nested radicals is

$$\{z_n\} = \sqrt[3]{6},\ \sqrt[3]{6 + (-1)\sqrt{6}},\ \sqrt[3]{6 + (-1)\sqrt{6 + \sqrt[3]{6}}},$$
$$\sqrt[3]{6 + (-1)\sqrt{6 + \sqrt[3]{6 + (-1)\sqrt{6}}}},\ \ldots,$$

which converges to $x \approx 1.48343452$. Thus, we have the equation

$$\sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \sqrt{6 + \sqrt[3]{6 - \cdots}}}}} \approx 1.48343452.$$

In 2014, Lynd proved the following convergence theorem which gives sufficient conditions for the convergence of a right nested radical [**4**].

Theorem 1. *Let $\{r_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{c_n\}$ and $\{a_n\}$ be periodic sequences of positive numbers.*
Then, the sequence corresponding to the right nested radical

$$\sqrt[r_1]{a_1 + c_1 \sqrt[r_2]{a_2 + c_2 \sqrt[r_3]{a_3 + c_3 \sqrt[r_4]{a_4 + \cdots}}}}$$

converges.

The next example shows that Theorem 1 does not apply to *left* nested radicals.

Example 2. Let the sequence of indices be the periodic sequence $\{r_n\} = 2, 3, 2, 3, \ldots,$ let the sequence of coefficients be the constant sequence $\{c_n\} = 1, 1, 1, \ldots,$ and let the sequence of radicands be the constant sequence $\{a_n\} = 22, 22, 22, \ldots.$

By Definition 1, the sequence of *right* nested radicals is

$$\sqrt{22},\ \sqrt{22 + \sqrt[3]{22}},\ \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}},\ \sqrt{22 + \sqrt[3]{22 + \sqrt{22 + \sqrt[3]{22}}}},\ \ldots,$$

which converges to five.

By Definition 1, the sequence of *left* nested radicals is

$$\{y_n\} = \sqrt{22},\ \sqrt[3]{22 + \sqrt{22}},\ \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}},\ \sqrt[3]{22 + \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}}},\ \ldots.$$

The subsequence $\{y_{2n+1}\}_{n=0}^{\infty}$ is

$$\sqrt{22}, \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}}, \sqrt{22 + \sqrt[3]{22 + \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}}}},\ \ldots,$$

which converges to five. The subsequence $\{y_{2n+2}\}_{n=0}^{\infty}$ is

$$\sqrt[3]{22 + \sqrt{22}}, \sqrt[3]{22 + \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}}},\ \sqrt[3]{22 + \sqrt{22 + \sqrt[3]{22 + \sqrt{22 + \sqrt[3]{22 + \sqrt{22}}}}}},\ \ldots,$$

which converges to three.

So, the sequence of left nested radicals $\{y_n\}$ does not have a limit. Instead, it asymptotically converges to the periodic sequence whose terms alternate between 5 and 3.

The first convergence theorem for left nested radicals was published in 1935 by A. Herschfeld. In [**2**], Herschfeld proved the following theorem.

Theorem 2. *For a nonnegative sequence $\{a_n\}$, a left nested radical of the form*

$$\cdots + \sqrt{a_5 + \sqrt{a_4 + \sqrt{a_3 + \sqrt{a_2 + \sqrt{a_1}}}}}$$

converges, if and only if there is an L such that $\{a_n\}$ converges to L.

Herschfeld's theorem applies to left nested radicals whose corresponding sequence of indices $\{r_n\}$ is a constant sequence of 2's and whose sequence of coefficients $\{c_n\}$ is a constant sequence of 1's. As the next example illustrates, neither part of this theorem holds in cases where $\{r_n\}$ or $\{c_n\}$ is nonconstant.

Example 3.

(a) The left nested radicals below converge to three even though their corresponding sequence of radicands $\{a_n\}$ is periodic and does not converge:

$$\cdots - \sqrt{6 + \sqrt{12 - \sqrt{6 + \sqrt{12 - \sqrt{6}}}}} = 3$$

$$\cdots + \sqrt{6 + \sqrt[3]{24 + \sqrt{6 + \sqrt[3]{24 + \sqrt{6}}}}} = 3.$$

(b) The left nested radical below does not converge even though its corresponding sequence of radicands $\{a_n\}$ is constant:

$$\cdots - \sqrt[3]{11 - \sqrt{11 - \sqrt[3]{11 - \sqrt{11 - \sqrt[3]{11}}}}}.$$

The corresponding sequence of nested radicals is

$$\sqrt[3]{11},\ \sqrt{11 - \sqrt[3]{11}},\ \sqrt[3]{11 - \sqrt{11 - \sqrt[3]{11}}},\ \sqrt{11 - \sqrt[3]{11 - \sqrt{11 - \sqrt[3]{11}}}},\ \ldots,$$

and the odd-indexed subsequence converges to two while the even-indexed subsequence converges to three.

We will investigate the class of left nested radicals of the form in Definition 1, where $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$ are periodic sequences.

In part 1 we present a technical lemma that will demonstrate how to relate left nested radicals to right nested radicals. This lemma allows us to utilize previously known theorems about right nested radicals.

In part 2 we use the lemma from part 1, combined with a theorem about right nested radicals, to prove three convergence results for a class of periodic left nested radicals.

In part 3 we prove two theorems that demonstrate how to construct a left nested radical with a predetermined end-behavior.

Part 1: Relating Left and Right Nested Radicals

We begin by defining a few terms from the field of difference equations.

Definition 2. Let $D \subseteq \mathbb{R}$, let $f : D \to D$, and let $x_1 \in D$.

(a) The equation

$$x_{n+1} = f(x_n) \textit{ for } n = 1,\ 2,\ 3,\ \ldots$$

is a first-order difference equation.

(b) The sequence

$$\{x_n\} = x_1,\ f(x_1),\ f(f(x_1)),\ \ldots,\ f^n(x_1),\ \ldots$$

is a solution of the difference equation.

(c) The corresponding equilibrium equation is $f(x) = x$ and any solutions of this equation are equilibrium points of the difference equation (these solutions are also referred to as fixed points of f).

The next theorem relates periodic right nested radicals to solutions of difference equations. In [**4**], this theorem was used to help prove several theorems about right nested radicals.

Theorem 3. *(Abbreviated version) Suppose that $\{r_n\}$ is a periodic sequence of integers that are greater than or equal to two, $\{c_n\}$ and $\{a_n\}$ are periodic sequences of real numbers, and the sequence $\{z_n\}$, corresponding to the right nested radical*

$$\sqrt[r_1]{a_1 + c_1 \sqrt[r_2]{a_2 + c_2 \sqrt[r_3]{a_3 + c_3 \sqrt[r_4]{a_4 + \cdots}}}}$$

is a sequence of real numbers. Let k be the least common multiple of the minimal periods of $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$.
Then, the following statements are true.

(a) The function

$$f(x) = \sqrt[r_1]{a_1 + c_1 \sqrt[r_2]{a_2 + \cdots + c_{k-1} \sqrt[r_k]{a_k + c_k x}}}$$

is continuous on a nonempty interval.

(b) The subsequences $\{z_{kn+1}\}_{n=0}^{\infty}$, $\{z_{kn+2}\}_{n=0}^{\infty}$, ..., and $\{z_{kn+k}\}_{n=0}^{\infty}$ are solutions of the difference equation

$$x_{n+1} = f(x_n) \text{ for } n = 1,\ 2,\ 3,\ \ldots,$$

with respective initial terms $x_1 = z_1 = \sqrt[r_1]{a_1}$ and for $j = 2, \ldots, k$, $x_1 = z_j = \sqrt[r_1]{a_1 + \cdots + c_{j-1} \sqrt[r_j]{a_j}}$.

(c) If the sequence $\{z_n\}$ converges to a limit L, then L is an equilibrium point of the difference equation in part (b).

Part (c) of Theorem 3 states that the limit of a periodic nested radical must be an equilibrium point of the corresponding difference equation. If the sequences $\{c_n\}$ and $\{a_n\}$ contain only rational numbers, then the equilibrium point is a root of a polynomial with rational coefficients. So, the only way that a periodic right (or left) nested radical can converge to π is if some of the coefficients or radicands are transcendental numbers.

The next example demonstrates how to apply Theorem 3.

Example 4. Let the sequence of indices be the periodic sequence $\{r_n\} = 3, 2, 3, 2, \ldots$, let the sequence of coefficients be the constant sequence $\{c_n\} = 1, 1, 1, \ldots$, and let the sequence of radicands be the periodic sequence $\{a_n\} = 5, 7, 5, 7, \ldots$.

By Definition 1, the sequence of *right* nested radicals is

$$\{z_n\} = \sqrt[3]{5},\ \sqrt[3]{5 + \sqrt{7}},\ \sqrt[3]{5 + \sqrt{7 + \sqrt[3]{5}}},\ \sqrt[3]{5 + \sqrt{7 + \sqrt[3]{5 + \sqrt{7}}}},\ \ldots.$$

The least common multiple of the periods of $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$ is two, so $k = 2$. The subsequence $\{z_{2n+1}\}_{n=0}^{\infty}$ is the solution of the difference equation

$$x_{n+1} = \sqrt[3]{5 + \sqrt{7 + x_n}} \text{ for } n = 1,\ 2,\ 3,\ \ldots,$$

where $x_1 = \sqrt[3]{5}$. The subsequence $\{z_{2n+2}\}_{n=0}^{\infty}$ is the solution of the same difference equation, but with the initial term $x_1 = \sqrt[3]{5 + \sqrt{7}}$.

Both subsequences converge to two. Checking that $2 = \sqrt[3]{5 + \sqrt{7 + 2}}$ verifies that two is an equilibrium point of the difference equation.

When analyzing a *left* nested radical, it is helpful to partition the sequence of nested radicals into subsequences, where each subsequence corresponds to a *right* nested radical. The following lemma was inspired by Theorem 3 and it relates periodic left nested radicals to periodic right nested radicals through their difference equations.

Lemma 1. *Suppose that $\{r_n\}$ is a periodic sequence of integers that are greater than or equal to two, $\{c_n\}$ and $\{a_n\}$ are periodic sequences of real numbers, and the sequence $\{y_n\}$, corresponding to the left nested radical*

$$\cdots + c_4 \sqrt[r_4]{a_4 + c_3 \sqrt[r_3]{a_3 + c_2 \sqrt[r_2]{a_2 + c_1 \sqrt[r_1]{a_1}}}}$$

is a sequence of real numbers. Let k be the least common multiple of the minimal periods of $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$.
Then, the following statements are true.

(a) In the case where $k = 1$, the left nested radical

$$\cdots + c_1 \sqrt[r_1]{a_1 + c_1 \sqrt[r_1]{a_1 + c_1 \sqrt[r_1]{a_1}}}$$

and the right nested radical

$$\sqrt[r_1]{a_1 + c_1 \sqrt[r_1]{a_1 + c_1 \sqrt[r_1]{a_1 + \cdots}}}$$

correspond to the same sequence of nested radicals.

(b) For $k \geq 2$, each function below is continuous on a nonempty interval.

$$f_1(x) = \sqrt[r_1]{a_1 + \underbrace{c_k \sqrt[r_k]{a_k + \cdots + c_1 x}}_{\text{decreasing subscripts}}},$$

$$f_j(x) = \sqrt[r_j]{\underbrace{a_j + \cdots\cdots + c_1}_{\text{decreasing subscripts}} \sqrt[r_1]{a_1 + \underbrace{c_k \sqrt[r_k]{a_k + \cdots + c_j x}}_{\text{decreasing subscripts}}}}, \text{ for } 1 < j < k,$$

$$f_k(x) = \sqrt[r_k]{\underbrace{a_k + \cdots\cdots + c_1}_{\text{decreasing subscripts}} \sqrt[r_1]{a_1 + c_k x}}.$$

(c) For each $j \in \{1, \ldots, k\}$, the subsequence $\{y_{kn+j}\}_{n=0}^{\infty}$ is the solution of the difference equation

$$x_{n+1} = f_j(x_n) \text{ for } n = 1, 2, 3, \ldots,$$

where $x_1 = y_j$.

(d) Each term in the subsequence $\{y_{kn+1}\}_{n=0}^{\infty}$ is a term in the sequence corresponding to the right nested radical

$$\sqrt[r_1]{a_1 + c_k \sqrt[r_k]{a_k + \cdots + c_1 \sqrt[r_1]{a_1 + \cdots}}}.$$

For $1 < j < k$, each term in the subsequence $\{y_{kn+j}\}_{n=0}^{\infty}$ is a term in the sequence corresponding to the right nested radical

$$\sqrt[r_j]{a_j + \cdots + c_1 \sqrt[r_1]{a_1 + c_k \sqrt[r_k]{a_k + \cdots + c_j \sqrt[r_j]{a_j + \cdots}}}}\,.$$

Each term in the subsequence $\{y_{kn+k}\}_{n=0}^{\infty}$ is a term in the sequence corresponding to the right nested radical

$$\sqrt[r_k]{a_k + \cdots + c_1 \sqrt[r_1]{a_1 + c_k \sqrt[r_k]{a_k + \cdots}}}\,.$$

(e) For $n \geq 1$,

$$\begin{aligned} y_{kn+1} &= \sqrt[r_1]{a_1 + c_k y_{kn}} \\ &\vdots \\ y_{kn+j} &= \sqrt[r_j]{a_j + c_{j-1} y_{kn+j-1}} \\ &\vdots \\ y_{kn+k} &= \sqrt[r_k]{a_k + c_{k-1} y_{kn+k-1}}\,. \end{aligned}$$

The proof of Lemma 1 has been omitted because it is similar to the proof of Theorem 3, which is given in [**4**].

The next example demonstrates how to apply Lemma 1 to a left nested radical. To help illustrate the similarities and differences between left and right nested radicals, this example uses the same sequences of indices, coefficients, and radicands as the right nested radical in Example 4.

Example 5. Let the sequence of indices be the periodic sequence $\{r_n\} = 3, 2, 3, 2, \ldots$, let the sequence of coefficients be the constant sequence $\{c_n\} = 1, 1, 1, \ldots$, and let the sequence of radicands be the periodic sequence $\{a_n\} = 5, 7, 5, 7, \ldots$.

By Definition 1, the sequence of *left* nested radicals is

$$\{y_n\} = \sqrt[3]{5},\ \sqrt{7+\sqrt[3]{5}},\ \sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}},\ \sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5}}}},\ \ldots\,.$$

The least common multiple of the periods of $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$ is two, so $k = 2$. The subsequence $\{y_{2n+1}\}_{n=0}^{\infty}$ is the solution of the difference equation

$$x_{n+1} = \sqrt[3]{5+\sqrt{7+x_n}} \quad \text{for } n = 1,\ 2,\ 3,\ \ldots,$$

where $x_1 = \sqrt[3]{5}$. The equilibrium point of this difference equation is two.

The subsequence $\{y_{2n+2}\}_{n=0}^{\infty}$ is the solution of the difference equation

$$x_{n+1} = \sqrt{7+\sqrt[3]{5+x_n}} \quad \text{for } n = 1,\ 2,\ 3,\ \ldots,$$

where $x_1 = \sqrt{7+\sqrt[3]{5}}$. The equilibrium point of this difference equation is three.

Each term in the subsequence $\{y_{2n+1}\}_{n=0}^{\infty}$ is a term in the sequence corresponding to the right nested radical

$$\sqrt[3]{5+\sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5+\cdots}}}}},$$

which converges to two.

Each term in the subsequence $\{y_{2n+2}\}_{n=0}^{\infty}$ is a term in the sequence corresponding to the right nested radical

$$\sqrt{7+\sqrt[3]{5+\sqrt{7+\sqrt[3]{5+\sqrt{7+\cdots}}}}},$$

which converges to three.

So, the sequence of left nested radicals $\{y_n\}$ asymptotically converges to the periodic sequence whose terms alternate between 2 and 3.

Part 2: Convergence Results

First, we consider left nested radicals whose coefficients and radicands are positive numbers.

Theorem 4. *Let $\{r_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{c_n\}$ and $\{a_n\}$ be periodic sequences of positive numbers. Let k be the least common multiple of the minimal periods of $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$. Let $\{y_n\}$ be the sequence corresponding to the left nested radical*

$$\cdots + c_3\sqrt[r_3]{a_3 + c_2\sqrt[r_2]{a_2 + c_1\sqrt[r_1]{a_1}}}\,.$$

Then, for each $j \in \{1, \dots, k\}$, the subsequence $\{y_{kn+j}\}_{n=0}^{\infty}$ converges to the unique equilibrium point of its corresponding difference equation.

Proof. By Lemma 1 part (d), each subsequence $\{y_{kn+j}\}$ is also a subsequence of a sequence of right nested radicals. By Theorem 1, each subsequence $\{y_{kn+j}\}$ converges. For each $j \in \{1, \dots, k\}$, let L_j be the limit of $\{y_{kn+j}\}$.

Each subsequence $\{y_{kn+j}\}$ is the solution of a difference equation of the form in Lemma 1 part (c), whose corresponding function f_j is of the form in Lemma 1 part (b). Since the terms in the sequences $\{c_n\}$ and $\{a_n\}$ are positive, one can show that each corresponding function

$$f_j(x) = \sqrt[r_j]{a_j + \cdots + c_1\sqrt[r_1]{a_1 + c_k\sqrt[r_k]{a_k + \cdots + c_j x}}}$$

is continuous, strictly increasing, and concave downward on the interval $[0, \infty)$.

Let $b = \max\{a_1, \dots, a_k, c_1, \dots, c_k, 2\}$ and let $M = b^2$. Since $b \geq 2$, we have $\sqrt[r]{b + bM} = \sqrt[r]{b + b^3} < \sqrt[r]{b^4} \leq b^2 = M$. So, for each $j \in \{1, \dots, k\}$,

$$\begin{aligned} f_j(0) &= \sqrt[r_j]{a_j + \cdots + c_k\sqrt[r_k]{a_k + \cdots + c_{j+2}\sqrt[r_{j+2}]{a_{j+2} + c_{j+1}\sqrt[r_{j+1}]{a_{j+1} + c_j \cdot 0}}}} \\ &> 0 \end{aligned}$$

and

$$
\begin{aligned}
f_j(M) &= \sqrt[r_j]{a_j + \cdots + c_k \sqrt[r_k]{a_k + \cdots + c_{j+2} \sqrt[r_{j+2}]{a_{j+2} + c_{j+1} \sqrt[r_{j+1}]{a_{j+1} + c_j M}}}} \\
&\le \sqrt[r_j]{b + \cdots + b \sqrt[r_k]{b + \cdots + b \underbrace{\sqrt[r_{j+2}]{b + b \underbrace{\sqrt[r_{j+1}]{b + bM}}_{< M}}}_{< M}}} \\
&\;\;\vdots \\
&< \sqrt[r_j]{b + bM} \\
&< M.
\end{aligned}
$$

We have shown that for each $j \in \{1, \ldots, k\}$, $f_j(0) > 0$, $f_j(M) < M$, and f_j is continuous, strictly increasing, and concave downward. Thus, each f_j has exactly one fixed point, so the corresponding difference equation for each subsequence $\{y_{kn+j}\}$ has exactly one equilibrium point. By Theorem 3 part (c), $L_1, \ldots, L_k$ are the unique equilibrium points of the difference equations corresponding to $\{y_{kn+1}\}, \ldots, \{y_{kn+k}\}$. ■

For a left nested radical of the form in Theorem 4, if each of its k corresponding difference equations has the same equilibrium point, then the sequence of nested radicals converges. However, if the corresponding difference equations do not all have the same equilibrium point, then the sequence of nested radicals converges to a periodic sequence whose minimal period divides k. This is the idea behind the proof of the next theorem which gives the necessary and sufficient conditions for the convergence of a class of left nested radicals.

Theorem 5. *Let $\{r_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{a_n\}$ be a periodic sequence of positive numbers. Let k be the least common multiple of the minimal periods of $\{r_n\}$ and $\{a_n\}$. Let $\{y_n\}$ be the sequence corresponding to the left nested radical*

$$\cdots + \sqrt[r_4]{a_4 + \sqrt[r_3]{a_3 + \sqrt[r_2]{a_2 + \sqrt[r_1]{a_1}}}}\,.$$

Then, $\{y_n\}$ converges to L, if and only if, $L > 1$ and for each $j \in \{1, \ldots, k\}$, $a_j = L^{r_j} - L$.

Proof. Suppose that $\{y_n\}$ converges to L. By taking the limit on each side of the equations in part (e) of Lemma 1, we get

$$
\begin{aligned}
L &= \sqrt[r_1]{a_1 + L} \\
L &= \sqrt[r_2]{a_2 + L} \\
&\;\;\vdots \\
L &= \sqrt[r_k]{a_k + L}.
\end{aligned}
$$

The equations above imply that $a_j = L^{r_j} - L$ for each $j \in \{1, \ldots, k\}$. Since each $a_j > 0$ and each $r_j \ge 2$, we know that L must be greater than one.

Suppose that $L > 1$ and for each $j \in \{1, \ldots, k\}$, $a_j = L^{r_j} - L$. By Theorem 4, each subsequence $\{y_{kn+j}\}$ converges to the unique equilibrium point of its corresponding difference equation, which is L (see below):

$$\begin{aligned}
f_1(L) &= \sqrt[r_1]{a_1 + \sqrt[r_k]{a_k + \cdots + \sqrt[r_3]{a_3 + \sqrt[r_2]{a_2 + L}}}} \\
&= \sqrt[r_1]{(L^{r_1} - L) + \cdots + \sqrt[r_3]{(L^{r_3} - L) + \sqrt[r_2]{(L^{r_2} - L) + L}}} \\
&= L \\
&\vdots \\
f_k(L) &= \sqrt[r_k]{a_k + \sqrt[r_{k-1}]{a_{k-1} + \cdots + \sqrt[r_2]{a_2 + \sqrt[r_1]{a_1 + L}}}} \\
&= \sqrt[r_k]{(L^{r_k} - L) + \cdots + \sqrt[r_2]{(L^{r_2} - L) + \sqrt[r_1]{(L^{r_1} - L) + L}}} \\
&= L.
\end{aligned}$$

Example 6. Let the sequence of indices be the periodic sequence $\{r_n\} = 2, 3, 4, 2, 3, 4, \ldots$, let the sequence of coefficients be the constant sequence $\{c_n\} = 1, 1, 1, \ldots$, and let the sequence of radicands be the periodic sequence $\{a_n\} = 2, 6, 14, 2, 6, 14, \ldots$.

By Definition 1, the sequence of *left* nested radicals is

$$\{y_n\} = \sqrt{2},\ \sqrt[3]{6 + \sqrt{2}},\ \sqrt[4]{14 + \sqrt[3]{6 + \sqrt{2}}},\ \sqrt{2 + \sqrt[4]{14 + \sqrt[3]{6 + \sqrt{2}}}},\ \ldots .$$

The least common multiple of the periods of $\{r_n\}$ and $\{a_n\}$ is three, so $k = 3$. Since $2 = 2^2 - 2$, $6 = 2^3 - 2$, and $14 = 2^4 - 2$, we know that $\{y_n\}$ converges to two. Notice that the conclusion in Herschfeld's theorem (Theorem 2) does not hold because the sequence of indices $\{r_n\}$ is not constant.

In Example 6, the corresponding sequence of right nested radicals is

$$\{z_n\} = \sqrt{2},\ \sqrt{2 + \sqrt[3]{6}},\ \sqrt{2 + \sqrt[3]{6 + \sqrt[4]{14}}},\ \sqrt{2 + \sqrt[3]{6 + \sqrt[4]{14 + \sqrt{2}}}},\ \ldots .$$

Other than the first term, $\{z_n\}$ does not contain any terms that are in $\{y_n\}$. However, $\{z_n\}$ also converges to two. This is not a coincidence, as the next corollary shows.

Corollary 1. *Let $\{r_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{a_n\}$ be a periodic sequence of positive numbers. Let $\{y_n\}$ be the sequence corresponding to the left nested radical*

$$\cdots + \sqrt[r_4]{a_4 + \sqrt[r_3]{a_3 + \sqrt[r_2]{a_2 + \sqrt[r_1]{a_1}}}}$$

and let $\{z_n\}$ be the sequence corresponding to the right nested radical

$$\sqrt[r_1]{a_1 + \sqrt[r_2]{a_2 + \sqrt[r_3]{a_3 + \sqrt[r_4]{a_4 + \cdots}}}}.$$

Then, whenever $\{y_n\}$ converges to L, $\{z_n\}$ converges to L.

Proof. Suppose that $\{y_n\}$ converges to L. Let k be the least common multiple of the periods of $\{r_n\}$ and $\{a_n\}$. By Theorem 5, $L > 1$ and for each $j \in \{1, \ldots, k\}$, $a_j = L^{r_j} - L$.

By Theorem 1, $\{z_n\}$ converges. By Theorem 3 parts (a) and (b), its corresponding difference equation is

$$x_{n+1} = f(x_n) \text{ for } n = 1,\ 2,\ 3, \ldots,$$

where $f(x) = \sqrt[r_1]{a_1 + \sqrt[r_2]{a_2 + \cdots + \sqrt[r_{k-1}]{a_{k-1} + \sqrt[r_k]{a_k + x}}}}$.

Using the argument in the proof of Theorem 4, it can be shown that this difference equation has a unique equilibrium point. Since $\{z_n\}$ converges, part (c) of Theorem 3 tells us that the limit of $\{z_n\}$ is the unique equilibrium point, which is L. (See below)

$$\begin{aligned} f(L) &= \sqrt[r_1]{a_1 + \sqrt[r_2]{a_2 + \cdots + \sqrt[r_{k-1}]{a_{k-1} + \sqrt[r_k]{a_k + L}}}} \\ &= \sqrt[r_1]{(L^{r_1} - L) + \cdots + \sqrt[r_{k-1}]{(L^{r_{k-1}} - L) + \sqrt[r_k]{(L^{r_k} - L) + L}}} \\ &= L. \end{aligned}$$

The conclusion of Corollary 1 does not hold if the nested radical has periodic coefficients. Example 1 shows that is possible for a pair of corresponding left and right nested radicals to have different limits.

Part 3: Constructing Left Nested Radicals

We now consider left nested radicals whose corresponding sequence of coefficients $\{c_n\}$ and sequence of radicands $\{a_n\}$ may contain negative numbers.

When proving the theorems in part 2, we relied on Theorem 1 to prove the convergence of the subsequences of a left nested radical. However, if there are negative coefficients or negative radicands, a right nested radical might not converge. For example, the right nested radical

$$\sqrt[3]{\frac{1}{2} - \sqrt[3]{\frac{1}{2} - \sqrt[3]{\frac{1}{2} - \sqrt[3]{\frac{1}{2} - \cdots}}}}$$

does not converge; its corresponding sequence of nested radicals asymptotically converges to a sequence with minimal period two. In this case, the equilibrium point of the corresponding difference equation is repelling.

In part 3, whenever we wish to prove that a sequence of right nested radicals converges, we utilize a special case of the Banach contraction principle.

Theorem 6. *(Special Case of the Banach Contraction Principle) [1, p. 58]*
Suppose that the following conditions hold:

(i) f *is continuous on* $[a, b]$

(ii) $f([a, b]) \subset [a, b]$

(iii) f' *exists and is continuous on* (a, b)

(iv) there exists k with $0 < k < 1$ such that $|f'(x)| < k$ for all $x \in (a, b)$.

Then, for all $x_1 \in [a, b]$, the corresponding solution of the difference equation

$$x_{n+1} = f(x_n) \text{ for } n = 1,\ 2,\ 3, \ldots$$

converges to the unique equilibrium point in $[a, b]$.

The next theorem was proven in [**4**] and applies to right nested radicals.

Theorem 7. *(Abbreviated version) Let $\{r_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{c_n\}$ be a periodic sequence where each $c_i \in \{-1, 1\}$. Let L be an integer that is greater than or equal to two.*
Then, there is a periodic sequence of positive integers $\{a_n\}$ such that

$$\sqrt[r_1]{a_1 + c_1 \sqrt[r_2]{a_2 + c_2 \sqrt[r_3]{a_3 + c_3 \sqrt[r_4]{a_4 + \cdots}}}} = L.$$

If all of the terms in the sequence $\{r_n\}$ are odd, then it is also true that

$$\sqrt[r_1]{-a_1 + c_1 \sqrt[r_2]{-a_2 + c_2 \sqrt[r_3]{-a_3 + c_3 \sqrt[r_4]{-a_4 + \cdots}}}} = -L.$$

Theorem 8 is the *left* nested radicals version of Theorem 7. Due to the nature of left nested radicals, its conclusion is stronger, stating that the sequence $\{a_n\}$ is unique.

The idea behind the proof of Theorem 8 is to first use Lemma 1 to partition the sequence of left nested radicals into subsequences that correspond to a right nested radical. Then, using the techniques in the proof of Theorem 7, show that each subsequence converges to L.

Theorem 8. *Let $\{r_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{c_n\}$ be a periodic sequence where each $c_i \in \{-1, 1\}$. Let L be a real number that is greater than or equal to two.*
Then, there is a unique periodic sequence of positive numbers $\{a_n\}$ such that

$$\cdots + c_4 \sqrt[r_4]{a_4 + c_3 \sqrt[r_3]{a_3 + c_2 \sqrt[r_2]{a_2 + c_1 \sqrt[r_1]{a_1}}}} = L.$$

If L is an integer, then $\{a_n\}$ must be a sequence of positive integers. If each term in the sequence $\{r_n\}$ is odd, then it is also true that

$$\cdots + c_4 \sqrt[r_4]{-a_4 + c_3 \sqrt[r_3]{-a_3 + c_2 \sqrt[r_2]{-a_2 + c_1 \sqrt[r_1]{-a_1}}}} = -L.$$

Proof. Let k be the least common multiple of the minimal periods of $\{r_n\}$ and $\{c_n\}$. Let $a_1 = L^{r_1} - c_k L$, and for each $j \in \{2, \ldots, k\}$, let $a_j = L^{r_j} - c_{j-1} L$. Thus, if L is an integer greater than or equal to two, then each a_j is a positive integer. Let $\{a_n\}$ be the period-k sequence that repeats the numbers $a_1, \ldots, a_k$. Let $\{y_n\}$ be the sequence of left nested radicals, as in Definition 1.

For each $j \in \{1, \ldots, k\}$, the subsequence $\{y_{kn+j}\}$ is the solution of a difference equation of the form in Lemma 1 part (c), whose corresponding function f_j is of the form in Lemma 1 part (b).

Using the techniques that were employed in the proof of Theorem 7, one can show that for each difference equation, the hypotheses of Theorem 6 are satisfied on the interval $[L-1, L+1]$. Thus, each subsequence $\{y_{kn+j}\}$ converges to the unique equilibrium point of its corresponding difference equation, which is L (see below):

$$\begin{aligned}
f_1(L) &= \sqrt[r]{a_1 + c_k\sqrt[r]{a_k + \cdots + c_3\sqrt[r]{a_3 + c_2\sqrt[r]{a_2 + c_1 L}}}} \\
&= \sqrt[r]{(L^r - c_k L) + \cdots + c_3\sqrt[r]{(L^r - c_2 L) + c_2\sqrt[r]{(L^r - c_1 L) + c_1 L}}} \\
&= L \\
&\vdots \\
f_k(L) &= \sqrt[r]{a_k + c_{k-1}\sqrt[r]{a_{k-1} + \cdots + c_2\sqrt[r]{a_2 + c_1\sqrt[r]{a_1 + c_k L}}}} \\
&= \sqrt[r]{(L^r - c_{k-1} L) + \cdots + c_2\sqrt[r]{(L^r - c_1 L) + c_1\sqrt[r]{(L^r - c_k L) + c_k L}}} \\
&= L.
\end{aligned}$$

To show that $\{a_n\}$ is unique, suppose there is a sequence $\{b_n\}$ with minimal period p such that

$$\cdots + c_4\sqrt[r_4]{b_4 + c_3\sqrt[r_3]{b_3 + c_2\sqrt[r_2]{b_2 + c_1\sqrt[r_1]{b_1}}}} = L.$$

Using the fact that kp is a multiple of the periods of $\{r_n\}$, $\{c_n\}$, and $\{b_n\}$, we take the limit on each side of the equations in part (e) of Lemma 1 and solve for the terms $b_1, \ldots, b_p$ to get

$$\begin{aligned}
L &= \sqrt[r_1]{b_1 + c_k L} && \Rightarrow b_1 = L^{r_1} - c_k L = a_1 \\
L &= \sqrt[r_2]{b_2 + c_1 L} && \Rightarrow b_2 = L^{r_2} - c_1 L = a_2 \\
&\vdots \\
L &= \sqrt[r_k]{b_k + c_{k-1} L} && \Rightarrow b_k = L^{r_k} - c_{k-1} L = a_k \\
L &= \sqrt[r_{k+1}]{b_{k+1} + c_k L} = \sqrt[r_1]{b_{k+1} + c_k L} && \Rightarrow b_{k+1} = L^{r_1} - c_k L = a_1 \\
&\vdots \\
L &= \sqrt[r_{kp}]{b_{kp} + c_{kp-1} L} = \sqrt[r_k]{b_p + c_{k-1} L} && \Rightarrow b_p = a_k \\
L &= \sqrt[r_{kp+1}]{b_{kp+1} + c_{kp} L} = \sqrt[r_1]{b_{p+1} + c_k L} && \Rightarrow b_{p+1} = a_1\ .
\end{aligned}$$

This shows that $\{b_n\}$ and $\{a_n\}$ are the same sequence.

To prove the last part of Theorem 8, where the nested radical converges to a negative number, use the same argument given in the proof of Theorem 7.

In the proof of Theorem 8, each subsequence of the nested radical was constructed to converge to L. It is possible to construct the subsequences so that they converge to different limits. In this case, the sequence of nested radicals would converge to a periodic sequence rather than a limit.

Theorem 9. *Let $\{L_n\}$ be a periodic sequence of integers that are greater than or equal to two. Let $\{c_n\}$ be a periodic sequence where each $c_i \in \{-1, 1\}$.*

Then, there is a positive integer r and a periodic sequence of positive integers $\{a_n\}$ such that the sequence corresponding to the left nested radical

$$\cdots + c_4\sqrt[r]{a_4 + c_3\sqrt[r]{a_3 + c_2\sqrt[r]{a_2 + c_1\sqrt[r]{a_1}}}}$$

converges asymptotically to the periodic sequence $\{L_n\}$.

Proof. Let k be the least common multiple of the minimal periods of $\{L_n\}$ and $\{c_n\}$. Let $m = \min\{L_1, \ldots, L_k\}$ and let $M = \max\{L_1, \ldots, L_k\}$. Since m and M are both greater than or equal to two, we can choose a positive integer r such that $m^r > M$. Let $a_1 = {L_1}^r - c_k L_k$, and for each $j \in \{2, \ldots, k\}$, let $a_j = {L_j}^r - c_{j-1}L_{j-1}$. Let $\{a_n\}$ be the period-k sequence that repeats the positive integers $a_1, \ldots, a_k$. Let $\{y_n\}$ be the sequence of left nested radicals, as in Definition 1.

For each $j \in \{1, \ldots, k\}$, the subsequence $\{y_{kn+j}\}$ is the solution of a difference equation of the form in Lemma 1 part (c), whose corresponding function f_j is of the form in Lemma 1 part (b).

Using the techniques that were employed in the proof of Theorem 7, one can show that for each difference equation, the hypotheses of Theorem 6 are satisfied on the interval $[L_j - 1, L_j + 1]$. Thus, each subsequence $\{y_{kn+j}\}$ converges to the unique equilibrium point of its corresponding difference equation. The corresponding equilibrium points are $L_1, \ldots, L_k$ (see below):

$$\begin{aligned}
f_1(L_1) &= \sqrt[r]{a_1 + c_k\sqrt[r]{a_k + \cdots + c_3\sqrt[r]{a_3 + c_2\sqrt[r]{a_2 + c_1 L_1}}}} \\
&= \sqrt[r]{({L_1}^r - c_k L_k) + \cdots + c_3\sqrt[r]{({L_3}^r - c_2 L_2) + c_2\sqrt[r]{({L_2}^r - c_1 L_1) + c_1 L_1}}} \\
&= L_1 \\
&\vdots \\
f_k(L_k) &= \sqrt[r]{a_k + c_{k-1}\sqrt[r]{a_{k-1} + \cdots + c_2\sqrt[r]{a_2 + c_1\sqrt[r]{a_1 + c_k L_k}}}} \\
&= \sqrt[r]{({L_k}^r - c_{k-1}L_{k-1}) + \cdots + c_2\sqrt[r]{({L_2}^r - c_1 L_1) + c_1\sqrt[r]{({L_1}^r - c_k L_k) + c_k L_k}}} \\
&= L_k.
\end{aligned}$$

We conclude the paper with the following example.

Example 7. Let $\{L_n\}$ be the periodic sequence that repeats the digits in the phone number 867-5309. Let the sequence of coefficients $\{c_n\}$ be the periodic sequence that repeats the seven numbers $1, 1, 1, 1, -1, 1, 1$. We will attempt to construct a sequence of left nested radicals that asymptotically converges to the periodic sequence $\{L_n\}$.

Since the sequence $\{L_n\}$ contains a zero, Theorem 9 does not apply. We can use the recipe given in the proof of Theorem 9 to construct a nested radical whose seven corresponding difference equations have the equilibrium points $L_1, \ldots, L_7$.

Let $r = 3$. Now, let $a_1 = {L_1}^r - c_7 L_7$, and for each $j \in \{2, \ldots, 7\}$, let $a_j = {L_j}^r - c_{j-1}L_{j-1}$. Then, $a_1 = 503$, $a_2 = 208$, $a_3 = 337$, $a_4 = 118$, $a_5 = 22$, $a_6 = 3$, and $a_7 = 729$. Let $\{a_n\}$ be the periodic sequence that repeats the seven numbers $a_1, \ldots, a_7$.

In [**8**] we demonstrate that the sequence corresponding to the left nested radical

$$\cdots + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503 + \sqrt[3]{729 + \sqrt[3]{3 - \sqrt[3]{22 + \sqrt[3]{118 + \sqrt[3]{337 + \sqrt[3]{208 + \sqrt[3]{503}}}}}}}}}}$$

appears to asymptotically converge to the periodic sequence $\{L_n\}$.

For each subsequence $\{y_{7n+j}\}$, the corresponding difference equation has the correct equilibrium point. However, when you examine the slope of the function in the difference equation, you see that it is undefined at the equilibrium point (because one of the L_n's is zero). This means that the equilibrium point is repelling and the solution of the difference equation will *not* converge to it. In fact, $\{y_n\}$ converges to a period-14 sequence that is *extremely* close to the period-7 sequence $\{L_n\}$, with $|y_n - L_n| < 10^{-15}$, for all $n > 28$.

Theorem 9 also fails to apply if the sequence $\{L_n\}$ contains a "1". Consider the sequence $\{L_n\} = 1, 2, 1, 2, \ldots$. If $\{c_n\} = 1, -1, 1, -1, \ldots$, then let let $r = 2$ and let $\{a_n\} = 3, 3, 3, \ldots$. Now, the corresponding sequence of left nested radicals $\{y_n\}$ asymptotically converges to $\{L_n\}$. However, if $\{c_n\} = 1, 1, 1, \ldots$, then there is no sequence $\{y_n\}$, of the form in Theorem 9, that asymptotically converges to $\{L_n\}$.

Future Research

Corollary 1 provided a sufficient condition for a pair of corresponding left and right nested radicals to converge to the same limit. However, the corollary does not hold if the sequence of coefficients is not constant.

For example, suppose that $\{r_n\} = 2, 2, 2, \ldots$ and $\{a_n\} = 6, 2, 6, 2, \ldots$. When $\{c_n\} = 1, -1, 1, -1, \ldots$, the left nested radical converges to two and the right nested radical is undefined (if a term in the sequence of nested radicals is nonreal, then we consider the nested radical to be undefined). When $\{c_n\} = -1, 1, -1, 1, \ldots$, the right nested radical converges to two and the left nested radical is undefined. And, when $\{c_n\} = 1, 2, 1, 2, \ldots$, the right nested radical converges, but the sequence of left nested radicals asymptotically converges to a period-2 sequence.

A problem for future research is to determine the necessary and sufficient conditions, in terms of $\{r_n\}$, $\{c_n\}$, and $\{a_n\}$, such that the sequence of left nested radicals $\{y_n\}$ and the sequence of right nested radicals $\{z_n\}$ are both defined and converge to the same limit.

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8. There is an online supplement to this paper that contains the numerical calculations for each example. The URL is: www.maa.org/sites/default/files/images/images/pubs/MathMagSupplements/15-00111MM_online%20supplement.pdf

Summary. We investigate several examples of *left* nested radicals and prove four convergence theorems. In the proofs of the theorems we provide a recipe for constructing nested radicals with a predetermined end-behavior. We conclude the paper by constructing a nested radical whose computed sequence becomes an endless repetition of the digits in the phone number 867-5309, just like in the song by Tommy Tutone.

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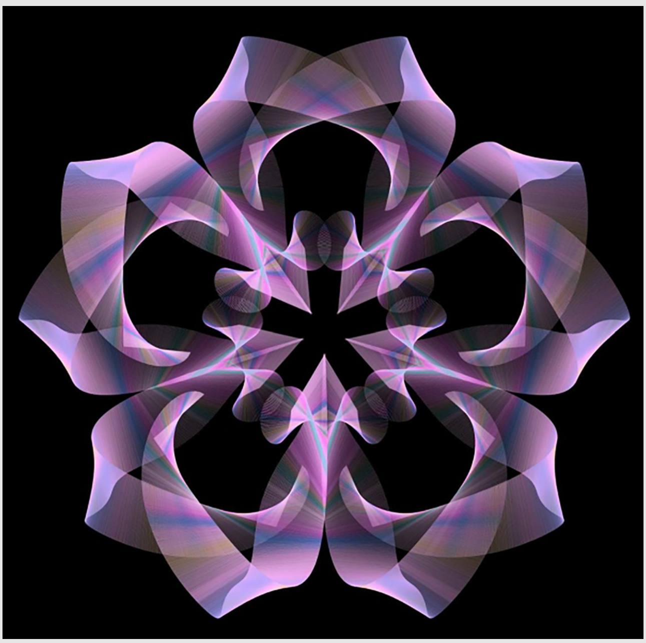

Artist Spotlight
Anne Burns

Complex Flow III, Anne Burns; digital print, 2010. Vectors whose length and direction are determined by a complex analytic function and whose color is a function of the slope of the vector are plotted along a path $y = f(x)$ and then reflected and rotated.

See interview on page 375.

Evaluation of Pi by Nested Radicals

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In this note we will show that

$$\pi = \lim_{k\to\infty} 2^{k+1}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \quad (\text{with } k+1 \text{ radicals}).$$

First, let a_n be the side length of a regular n-polygon inscribed in a unit circle where n is an integer ≥ 3 and O is the center of the circle. To calculate a_n, let's consider a triangle formed by a_n and two radii; see Figure 1(a). By the law of cosines we get ${a_n}^2 = 2 - 2\cos(2\pi/n) = 2 - 2(1 - 2\sin^2(\pi/n))$, so $a_n = 2\sin(\pi/n)$.

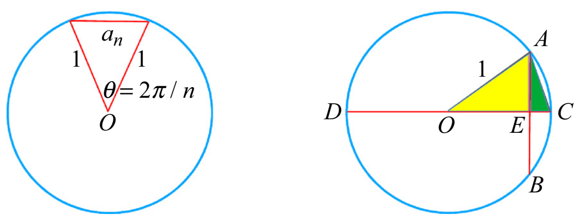

Figure 1 (a) Finding a_n. (b) Finding a_{2n}.

Theorem. $a_{2n} = \sqrt{2-\sqrt{4-{a_n}^2}}$ *for any integer* $n \geq 3$.

Proof. For convenience, let a_n be $|\overline{AB}|$, i.e., the length of the line segment $\overline{AB}$. Let $\overline{CD}$ be the diameter perpendicular to $\overline{AB}$ at the point E; see Figure 1(b). Note that $|\overline{AC}| = a_{2n}$. In the right triangle ΔAEO, the length $|\overline{OE}|$ is $\sqrt{1-(a_n/2)^2}$ by the pythagorean theorem. Similarly, $|\overline{EC}| = \sqrt{{a_{2n}}^2 - (a_n/2)^2}$ in ΔAEC. Since $|\overline{OE}| + |\overline{EC}| = 1$, we have $\sqrt{{a_{2n}}^2 - ({a_n}^2/4)} = 1 - \sqrt{1-({a_n}^2/4)}$. After squaring both sides of this equation, we obtain ${a_{2n}}^2 = 2 - 2\sqrt{1-({a_n}^2/4)}$, then the result follows. ∎

Moreover, by mathematical induction on k, we can prove that $a_{2^k n} = \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{4-{a_n}^2}}}}}$ (with $k+1$ radicals), which is equal to $\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos(\pi/n)}}}}$ (with k radicals), since $a_n = 2\sin(\pi/n)$.

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MSC: Primary 00A05

Because $a_{2^k n}$ is the side length of a regular $2^k n$-polygon inscribed in a unit circle, the perimeter of the polygon approaches the circumference of the circle, so that

$$\lim_{k\to\infty}(2^k n \cdot a_{2^k n}) = 2\pi, \quad \text{or} \quad \lim_{k\to\infty}\left(2^k \cdot \frac{n}{2} \cdot a_{2^k n}\right) = \pi \text{ as desired.}$$

By setting $n = 4$ in the last equation, we have proved the limit given at the beginning.

Remark. We modified the proof in [**1**] and obtained a different result.

REFERENCE

1. G. Guo, *Chudeng Shuxue Yanjiu* (in Chinese). Harbin Institute of Technology Press, 2008. 537–538.

Summary. We use nested radicals to represent pi as a limit.

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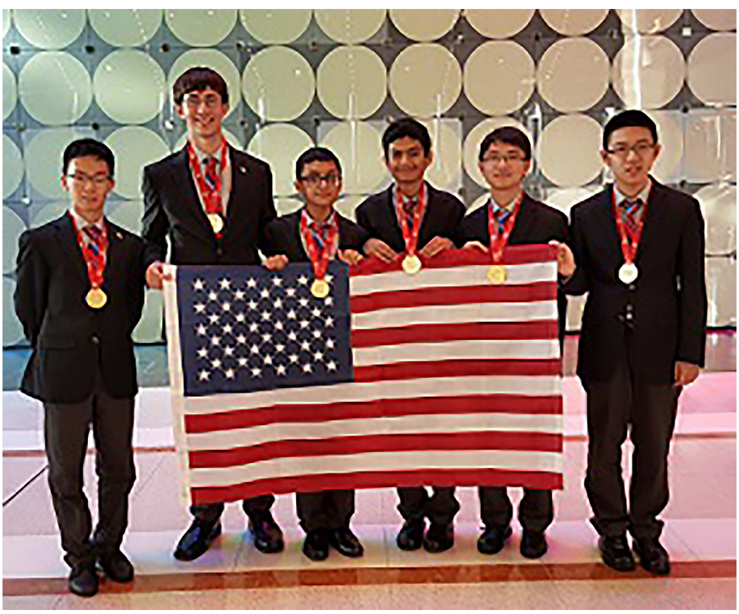
U.S. International Mathematical Olympiad Team



Members of the winning 2016 U.S. team were Ankan Bhattacharya, Michael Kural, Allen Liu, Junyao Peng, Ashwin Sah, and Yuan Yao, all of whom were awarded gold medals for their individual scores. Photo by team coach, Po-Shen Loh from Carnegie Mellon University.

Proof Without Words: Factorial Sums

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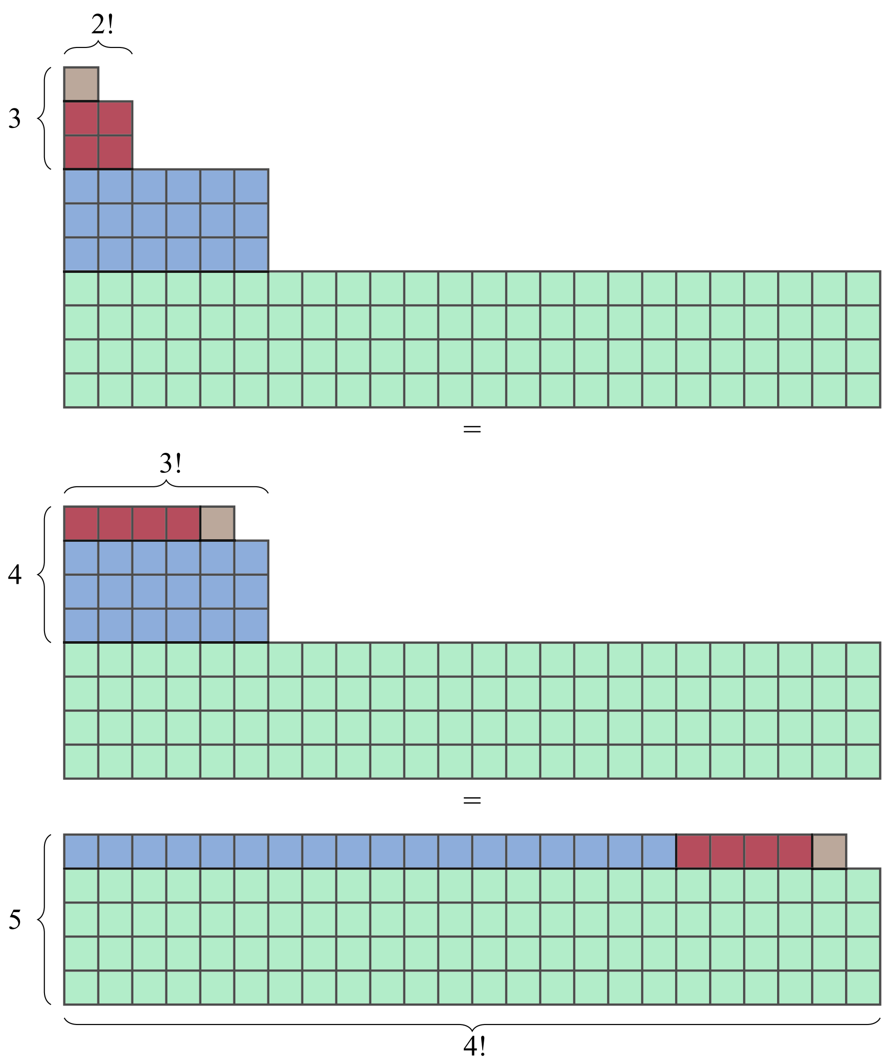
Theorem. *For any integer $n \geq 1$, we have* $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1.$

Proof. (e.g., for $n = 4$).



Corollary. *For each positive integer m, there exists a unique sequence of integers $\{a_i\}_{i\geq 1}$, satisfying $0 \leq a_i \leq i$ for all i, such that $m = \sum_{i\geq 1} a_i \cdot i!$.*

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MSC: Primary 05A10

Remark. Given such a representation of m, the theorem describes how to produce a representation of $m+1$; the corollary follows by induction (since $1 = 1 \cdot 1!$).

Summary. We provide a visual proof of a factorial sum identity implying the existence of the factorial base number system.

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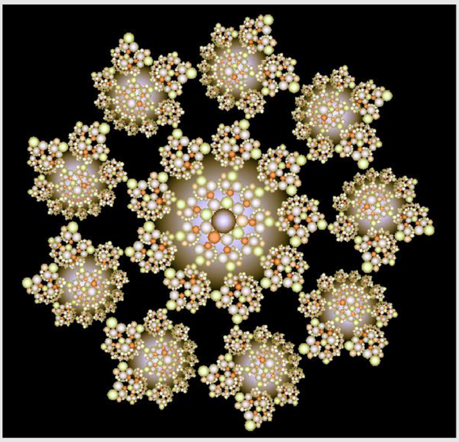

Artist Spotlight
Anne Burns

Nine plus one, Anne Burns; digital print, 2005. An iterated function system made up of Mobius transformations that map the unit disk to a chain of nine disks plus a disk in the center. The nine disks are pairwise tangent, tangent to the unit circle, and tangent to the tenth disk in the center of the unit circle.

See interview on page 375.

L	A	F	F			B	A	B	E		M	A	K	E
I	L	I	A	D		A	R	O	D		A	L	E	S
S	I	L	V	E	R	B	E	R	G		T	I	N	T
A	T	L	A	N	T	A		G	E	O	R	G	I	A
				S	E	R	F			M	I	N	G	
A	N	N	I	E	S		R	O	L	E	X			
L	A	O	S				A	J	O	G		A	M	S
U	P	A	R	R	O	W	N	O	T	A	T	I	O	N
M	A	A		O	P	E	C				I	D	E	O
			T	U	T	T	I		N	A	G	A	N	O
	T	W	I	G			S	E	C	T				
N	O	E	T	H	E	R		T	A	A	L	M	A	N
E	T	A	L		C	A	N	T	A	R	E	L	L	A
W	A	V	E		H	I	Y	A		I	N	L	A	Y
S	L	E	D		O	N	E	S			T	E	N	S

ACROSS

1. TV network airing sitcom reruns, or a Univ. Texas MOOC that teaches a standard math subject using MATLAB
5. The pig in "That'll do, pig"
9. "Three and four ___ seven"
13. Story about Troy
15. Poster ballplayer for steroids: Abbr.
16. Drinks that come in pale and brown varieties
17. * Alice from University of California who will deliver an AMS-MAA invited address about cryptography
19. Hue
20. * With 21-Across, location of JMM 2017
21. * See 20-Across
23. Feudal laborer
25. Former Chinese NBA star Yao
26. Organic food brand whose logo features a rabbit
30. Swiss luxury watch brand
33. Neighbor of Thailand and Vietnam
34. "I'm going out for ___" (2 wds.)
35. * Co-host of JMM 2017
38. Invention of Donald Knuth that concisely expresses exponentiation, tetration, and further generalizations
43. * Co-host of JMM 2017
44. Oil cartel
45. Prefix with logical
46. ___ -frutti
47. Japanese city that hosted the 1998 Winter Olympics
49. Tiny bit of a branch
51. Religious offshoot
53. * Mathematician Emmy, eponym for a lecture sponsored by the Association for Women in Mathematics
56. * Laura from James Madison University who will give an invited address about 3D printing
61. And others, for short, often used when citing a paper
62. * Jason from University of Georgia who will give an invited address about random polygons
64. * "Focusing energy critical ___ equation": subject of 11-Down's invited lectures
65. "Hey!"
66. Embed within
67. Iditarod vehicle
68. Digit furthest to the right
69. Digit second-most furthest to the right

DOWN

1. * Jeffrey of the University of Toronto who will give the 53-Across lecture on symplectic manifolds
2. Came down to earth
3. What a curve may do to a space (e.g., Peano or Hilbert curve)
4. Type of bean (that pairs well with a nice chianti?)
5. Children's book elephant character
6. "___ we having fun yet?"
7. Former Wimbledon champion Bjorn
8. Connection between vertices
9. A square one has a trace
10. Arrange
11. * Carlos of the University of Chicago who will give three invited colloquium lectures
12. "Cômo ___?"
14. Like a graph with close to the maximum number of 8-Downs
18. I-95 and U.S. 101, e.g.
22. Smallest infinite ordinal number
24. * Su who will deliver the 43-Across Retiring Presidential Address
26. Terence Tao vis-a-vis Princeton, or Andrew Wiles vis-a-vis Oxford
27. California wine valley
28. U.S. org. that conducts environmental and climate research
29. Home of Noga Alon and Saharon Shelah: Abbr.
31. Eye in Spanish
32. Parking place
35. Opera set in Egypt
36. Faucet brand
37. Name of the alien in Reddit's logo
39. ___ sketch: an idea of a proof
40. Select, with "for"
41. Sopping
42. Standup comedian Notaro
46. Gave a name to one's talk
47. March Madness org.
48. Pioneering video game company
49. Add up
50. Interlace
52. Singer James and namesakes
53. What CNN purports to provide
54. Audio bounce-back
55. Picnic hamperer
57. It comes after Mardi Gras
58. Ms. in French: Abbr.
59. Mathematician Turing
60. Thumbs-down votes
63. The Science Guy

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Joint Math Meetings 2017

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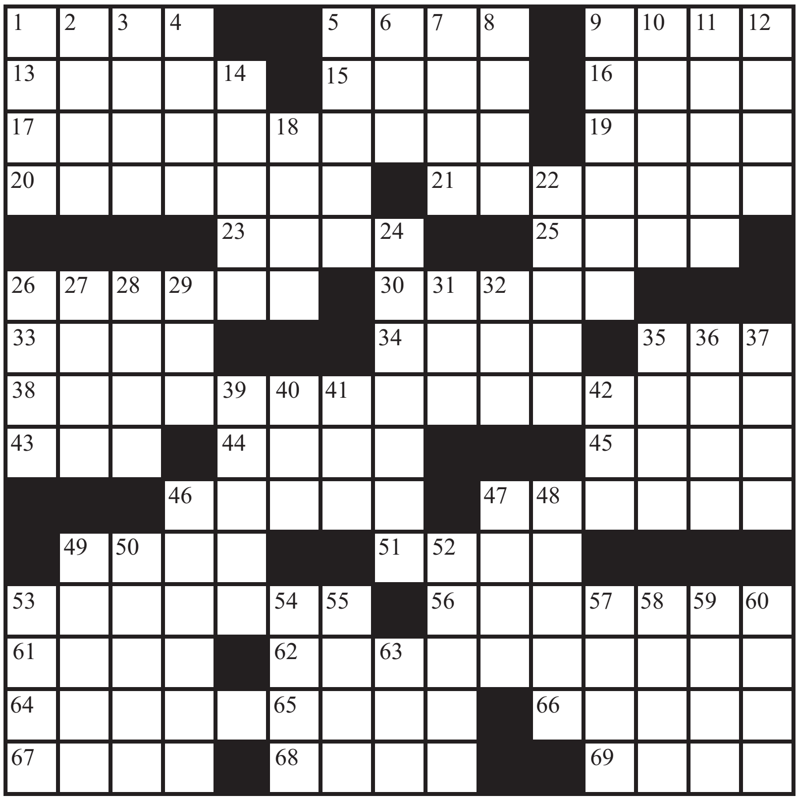

Clues start at left, on page 340. The Solution is on page 339.

Extra copies of the puzzle can be found at the MAGAZINE's website, www.maa.org/mathmag/supplements.

Crossword Puzzle Creators

If you are interested in submitting a mathematically themed crossword puzzle for possible inclusion in MATHEMATICS MAGAZINE, please contact the editor at mathmag@maa.org.

Fibonacci, Lucas, and a Game of Chance

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Imagine you are one of n people seated around a table playing a game of chance. To your immediate left sits Leonardo of Pisa, and to your right, Édouard Lucas. There is a \$100 bill in front of you. At each turn the bill moves one position to the left with probability $1/3$, one position to the right with probability $1/3$, and with probability $1/3$ the game ends and the current bill-holder wins. What is the probability that you win the game?

It may not be immediately clear why Fibonacci and Lucas are part of this game. We hope to convince you that they deserve a seat at the table.

Background: *LCR*

The game of chance outlined above is a simplified version of the popular dice game *Left Center Right*, or *LCR*, published by George & Company, LLC in 1992. *LCR* is a game of pure chance for three or more players. Each player begins with three chips. There are also three dice: each is six-sided with three faces marked L, C, and R, and the remaining three faces marked with a dot. The first player rolls the three dice. For each L rolled, he passes one of his chips to the player on his left. For each R rolled, he passes one of his chips to the player on his right. And for each C rolled, he passes a chip to a pot in the center of the table. For each dot rolled, nothing happens. The dice are then passed to the left, and the next player rolls. If a player has three or more chips, he rolls all three dice. Otherwise he rolls only as many dice as he has chips. In particular, if a player has no chips, play passes to his left (but he is still in the game, for he may still acquire chips from his neighbors). Since chips can never leave the center position, eventually all chips accumulate there. According to the official rules, the game ends when only one player has chips, and that player is the winner.

The game *LCR Wild*, introduced in 2013, replaces some of the dots on the dice with the word "wild." For each *wild* that a player rolls, he takes a chip from another player of his choosing. This introduces some strategy into the game. The original *LCR* is a game of pure chance; no player makes any strategic decisions at any point during the game. A tongue-in-cheek account of this observation can be found at [**2**].

The Endgame

Our game of chance with the \$100 bill is a simple variant of the standard *LCR* endgame. Typically, a game of *LCR* reaches a state where only one chip remains on

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MSC: Primary 91A60; Secondary 11B39

the table. While the official rules declare the chip-holder to be the winner, our variant requires that play continue; *the game ends when the last chip is moved to the center, and the person who puts it there is the winner*.

When there is only one chip left in play some special circumstances apply. First, note that only one die is rolled per turn for the remainder of the game, and the holder of the chip is the only player who rolls. And since rolling a dot when there is only one chip in play changes nothing (the player who rolled it simply rolls again), we may think of the die as being three-sided with three equally likely outcome: L, C, and R. Another way to see this is as follows: To calculate the probability that the first move the chip makes is to the left, one sums over all nonnegative integers k the probability of rolling k dots followed by an L. This is a geometric series:

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\frac{1}{6}=\left(\frac{1}{1-1/2}\right)\frac{1}{6}=\frac{1}{3}.$$

So when playing this endgame with the last chip, the only possible outcome for a single turn is L, C, or R, each with probability 1/3. In other words, *LCR* played with this endgame eventually becomes identical to our game of chance with the \$100 bill. One can make the case that this endgame extends gameplay in a natural way, but more importantly, it leads to our central question: When only one chip remains, what is the probability that the player holding the chip wins the game?

Enter Fibonacci and Lucas

Recall that the Fibonacci numbers F_n are the terms of the sequence 0, 1, 1, 2, 3, 5, . . ., where $F_0 = 0$ and $F_1 = 1$, and each successive term is the sum of the previous two. The Lucas numbers L_n follow the same second-order recurrence, but begin $L_0 = 2$ and $L_1 = 1$, so the Lucas sequence begins 2, 1, 3, 4, 7, 11, 18. Each sequence can be extended to utilize negative indices by running the recurrence in reverse. For instance, $F_{-1} + F_0 = F_1$, so $F_{-1} = F_1 - F_0 = 1$. It is a simple exercise to show for odd n that $F_{-n} = F_n$, and for even n that $L_{-n} = L_n$.

Now the likelihood that the bill-holder wins our game of chance depends, of course, on the number of players seated around the table. Suppose there are n players. We will prove the following delightful result: If n is odd, the bill-holder wins with probability F_n/L_n, and if n is even, the bill-holder wins with probability $L_n/(5F_n)$. In fact, we will show that each player's winning probability can be expressed as a simple ratio with Fibonacci and Lucas numbers.

Equally intriguing is the analogous question when there are infinitely many players. In that case, the probability that the bill-holder wins is $1/\sqrt{5}$, and the other players' winning probabilities are easily derived from this.

It is a simple matter to verify that if A_n is any sequence satisfying the recurrence $A_{n+1} = A_n + A_{n-1}$, then

$$A_{n+2} = 3A_n - A_{n-2}. \tag{1}$$

Hence this recurrence is satisfied by both the Fibonacci and Lucas numbers.

Three well-known relations between the Fibonacci and Lucas numbers will be useful (see equations 13, 17b, 17c, and 24 in [**5**], for example):

$$F_{2n} = F_n L_n \tag{2}$$

$$L_{2n} - 2 = \begin{cases} L_n^2 & \text{for } n \text{ odd} \\ 5F_n^2 & \text{for } n \text{ even} \end{cases} \tag{3}$$

$$\lim_{n\to\infty} L_n/F_n = \sqrt{5}. \tag{4}$$

Recurrence and Symmetry

In our game of chance with infinitely many players, label the positions of the players with the integers. If instead there are n players, label their positions 0 through $n-1$ in a counterclockwise progression when viewed from above. In either case, assume that the bill begins at position 0. When the number of players is understood, or when there are infinitely many players, we let P_k denote the probability that the player in position k wins the game. When we wish to emphasize that a game has n players, we write $P_{n,k}$ instead of P_k. In the case of n players, we will sometimes find it convenient to write P_n for P_0.

We begin with a fundamental recurrence relation in the following lemma.

Lemma 1. *For an n-player game with* $0 < k < n$*, or if there are infinitely many players and* $k > 0$*, we have* $P_{k+1} = 3P_k - P_{k-1}$.

Proof. Calculate P_k by conditioning on the first move. With probability $1/3$ the bill's first move is to the left. In this case the bill now must move $k+1$ places to the right in order to reach the player in position k, rather than k places. Hence player k's likelihood of winning from this position is P_{k+1}. Similarly, with probability $1/3$ the bill first moves right, and in this case the bill now must move $k-1$ places to the right in order to reach the player in position k. Hence player k's likelihood of winning from this position is P_{k-1}. And of course, there is a $1/3$ chance that the player in position 0 wins on the first move, in which case the player in position k cannot win (since $k \neq 0$ and $k \neq n$). Hence we have

$$P_k = \frac{1}{3}P_{k+1} + \frac{1}{3}P_{k-1} + 0,$$

from which the result follows. ∎

This second-order recurrence means that if we know P_0 and P_1, we can find every other P_k. The next lemma tells us that we really only need P_0.

Lemma 2. *Whether there are finitely many or infinitely many players, we have*

$$2P_1 = 3P_0 - 1.$$

Proof. In the case of infinitely many players, left-right symmetry gives $P_{-1} = P_1$. And in the case of n players we have $P_{n-1} = P_1$ (indeed, we have $P_{n-k} = P_k$ for all $0 \le k \le n$). Now, as in the previous lemma, calculate P_0 by conditioning on the first move: With probability $1/3$ the bill's first move is to the left. In this case the original bill-holder is now adjacent to the current bill-holder, so the likelihood that the original bill-holder wins from this position is P_1. Similarly, with probability $1/3$ the bill first moves right, and in this case the original bill-holder again wins with probability P_1. And of course, there is a $1/3$ chance that the original bill holder wins on the first move. Hence we have

$$P_0 = \frac{1}{3}P_1 + \frac{1}{3}P_1 + \frac{1}{3},$$

from which the result follows. ∎

Now it is a simple matter to derive an explicit formulation for any P_k in terms of P_0.

Theorem 1. *For an n-player game with $0 \le k \le n$, or if there are infinitely many players and $k \ge 0$, we have $2P_k = L_{2k}P_0 - F_{2k}$.*

Proof. We use induction on k, with the $k = 0$ case being trivial, and with Lemma 2 establishing the case $k = 1$. We then let $k \ge 2$ and assume that $2P_j = L_{2j}P_0 - F_{2j}$ for $j < k$. Then Lemma 1 gives

$$\begin{aligned}
2P_k &= 2(3P_{k-1} - P_{k-2}) \\
&= 3(2P_{k-1}) - 2P_{k-2} \\
&= 3(L_{2k-2}P_0 - F_{2k-2}) - (L_{2k-4}P_0 - F_{2k-4}) \\
&= (3L_{2k-2} - L_{2k-4})P_0 - (3F_{2k-2} - F_{2k-4}) \\
&= L_{2k}P_0 - F_{2k},
\end{aligned}$$

where the last equality follows from equation 1. ∎

Finitely Many Players

Let us now focus on a game with n players, and write $P_{n,k}$ for P_k. Theorem 1 gives $2P_{n,n} = L_{2n}P_{n,0} - F_{2n}$, and since $P_{n,n} = P_{n,0}$, we immediately obtain player zero's winning probability as

$$P_{n,0} = \frac{F_{2n}}{L_{2n} - 2},$$

which by equations 2 and 3 can be expressed

$$P_{n,0} = \begin{cases} \frac{F_n}{L_n} & \text{for } n \text{ odd} \\ \frac{L_n}{5F_n} & \text{for } n \text{ even.} \end{cases} \tag{5}$$

This gives rise to a beautiful expression for *each* player's winning probability.

Theorem 2. *In our game of chance with n players, we have for $0 \le k \le n$*

$$P_{n,k} = \begin{cases} \frac{F_{n-2k}}{L_n} & \textit{for n odd} \\ \frac{L_{n-2k}}{5F_n} & \textit{for n even.} \end{cases}$$

Proof. We note that for $k > n/2$, the indices on the Fibonacci and Lucas numbers in the numerators will be negative. This poses no problems, and indeed it ensures that $P_{n,k} = P_{n,n-k}$.

Suppose n is odd, say $n = 2m + 1$. Let us move across the table from position 0 to the players in positions m and $m + 1$. Each is equidistant from position 0, so symmetry dictates that $P_{n,m+1} = P_{n,m}$. If we guess that each of these probabilities is equal to $1/L_n$ and note that $F_{-1} = F_1 = 1$, we see that two adjacent players have winning probabilities proportional to two consecutive odd-indexed Fibonacci numbers. The recurrence for the odd-indexed Fibonacci numbers (equation 1) is identical to the recurrence of Lemma 1, so it follows that $P_{n,m-1} = F_3/L_n$, $P_{n,m-2} = F_5/L_n$, all the way down to the correct value $P_{n,0} = F_n/L_n$ (equation 5). Since the value of $P_{n,0}$ uniquely determines every other $P_{n,k}$, our guess must be correct. Hence we have established that $P_{n,k} = F_{n-2k}/L_n$.

If n is even, say $n = 2m$, the proof follows a similar track, with a few subtle differences. Symmetry now dictates that $P_{n,m+1} = P_{n,m-1}$. By Lemma 1 we obtain $3P_{n,m} = 2P_{n,m-1}$. Now we guess that the player in position m, opposite the bill-holder, has winning probability $P_{n,m} = 2/(5F_n) = L_0/(5F_n)$. This implies

$$P_{n,m-1} = \frac{3}{2}P_{n,m} = \frac{3}{2}\frac{2}{5F_n} = \frac{3}{5F_n} = \frac{L_2}{5F_n},$$

so two adjacent players have winning probabilities proportional to L_0 and L_2. The recurrence for the even-indexed Lucas numbers (equation 1) is identical to the recurrence of Lemma 1, so it follows that $P_{n,m-2} = L_4/(5F_n)$, $P_{n,m-3} = L_6/(5F_n)$, all the way down to the correct value $P_{n,0} = L_n/(5F_n)$ (equation 5). Since the value of $P_{n,0}$ uniquely determines every other $P_{n,k}$, our guess must be correct. This establishes that $P_{n,k} = L_{n-2k}/(5F_n)$. ■

Figure 1 shows the probabilities $P_{n,k}$ for $n = 10$ and $n = 11$. Observe that when n is even, the player seated opposite the bill-holder has winning probability $2/(5F_n)$, and running down either side of the table, the numerators are the even-indexed Lucas numbers. When n is odd, the two players opposite the bill-holder each have winning probability $1/L_n$, and running down either side of the table, the numerators are the odd-indexed Fibonacci numbers.

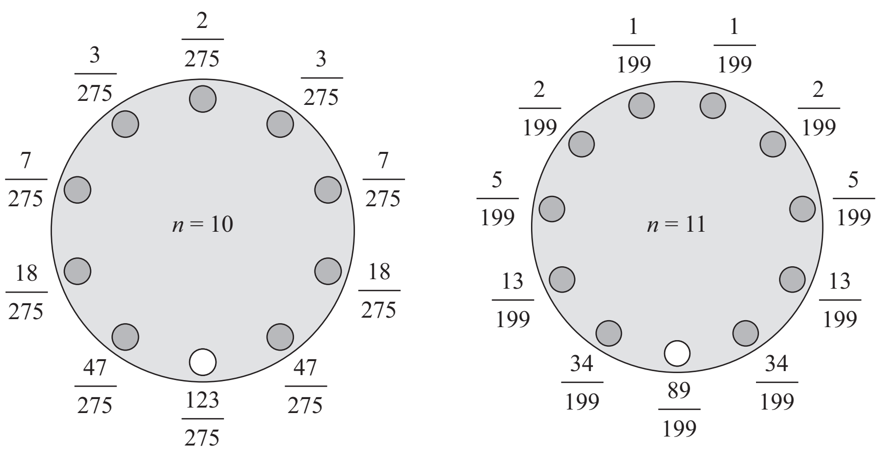

Figure 1 Probabilities $P_{n,k}$ for $n = 10$ and 11. The bill-holder (in position 0) is at the bottom of the table.

We note also that whether n is odd or even, the recurrence of Lemma 1 provides $n - 1$ linear equations (for $P_{n,2}$ through $P_{n,n}$), while Lemma 2 and $P_{n,0} = P_{n,n}$ provide two additional linear equations. So it is certainly possible to set up and solve an $(n + 1) \times (n + 1)$ system of linear equations in order to prove Theorem 2. And in case you're counting, there is an additional linear equation that we did not consider: The sum of the n probabilities must be one! Indeed, in both the odd and even cases Theorem 2 together with the fact that the probabilities sum to one imply a (well-known) relationship between Fibonacci and Lucas numbers.

Infinitely Many Players

When there are infinitely many players, we will henceforth write p_k instead of P_k for the probability of a win for the player in position k, in order to emphasize that we are no longer in the finite case. Because left and right moves are equally likely, $p_k = p_{-k}$, so it suffices to find p_k for $k \geq 0$.

Now Theorem 1 still applies, so every p_k will be known if we can determine p_0. But as the number of players gets large, it becomes vanishingly unlikely that the bill will complete a full revolution around the table. Hence we have

$$p_0 = \lim_{n\to\infty} P_{n,0}.$$

Theorem 2 gives different expressions for $P_{n,0}$ according to whether n is even or odd, but in either case, equation 4 can be used to obtain a common limiting value:

$$p_0 = \frac{1}{\sqrt{5}} \approx 0.44721. \tag{6}$$

We note that p_0 may also be calculated directly, forgoing any consideration of a game with finitely many players. Briefly: First calculate the probability that the player in position 0 wins *and* that the bill moves m times prior to the winning move. In order that the initial sequence of left and right moves return the bill to position 0, there must be an equal number of left moves and right moves. So for some $k \geq 0$ there are k Ls and k Rs in the initial move sequence. There are $\binom{2k}{k}$ such sequences. Each move has probability $1/3$, as does the final winning move, and moves are independent of one another. Therefore, the probability that the bill moves $m = 2k$ times and the player in position 0 wins is

$$\binom{2k}{k}\frac{1}{3^{2k+1}}.$$

It follows that

$$p_0 = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{3^{2k+1}}.$$

This series can be evaluated by finding its generating function (see, for example, [**7**]). That is, one can show that the formal power series

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{3^{2k+1}}x^k$$

is the Maclaurin series for the function

$$f(x) = \frac{1}{\sqrt{9-4x}},$$

with radius of convergence $9/4$. And since the radius of convergence is greater than 1, we conclude that $p_0 = f(1) = 1/\sqrt{5}$.

With p_0 known, Theorem 1 immediately gives the simple recursive expression

$$p_k = \frac{1}{2}\left(\frac{1}{\sqrt{5}}L_{2k} - F_{2k}\right). \tag{7}$$

Moreover, a beautiful closed-form expression for p_k is easily obtained.

Theorem 3. *In a game with infinitely many players and $k \geq 0$,*

$$p_k = p_0\lambda^k = \frac{1}{\sqrt{5}}\lambda^k,$$

where $\lambda = \frac{3-\sqrt{5}}{2} \approx 0.38197$ is the squared reciprocal of the golden ratio.

Proof. Lemma 2 gives

$$p_1 = \frac{1}{2}(3p_0 - 1) = p_0\frac{3-\sqrt{5}}{2} = p_0\lambda.$$

The recurrence $p_{k+1} = 3p_k - p_{k-1}$ of Lemma 1 has characteristic polynomial $t^2 - 3t + 1$. This polynomial has two distinct roots, λ and $1/\lambda$, and since we know the initial values p_0 and p_1, a straightforward calculation gives the result. ■

We note that Theorem 3 may also be proved from equation 7 by writing the Binet formulae for L_{2k} and F_{2k}, and noting that $\lambda = \phi^{-2}$, where ϕ is the golden ratio.

In the infinite setting it will be important to find the probability that the bill, originally at position 0, eventually arrives at position k. Let us formalize this idea: For $k > 0$, let E_k be the event that there exists a positive integer m such that after the initial m movements (starting at position 0) the bill is at position k. We call the event E_k the event that "the bill arrives at position k." Note that in order for the player at position k to win the game, the event E_k must occur. Therefore,

$$\begin{aligned} p_k &= P(\text{player } k \text{ wins}) \\ &= P(\text{player } k \text{ wins } \cap\ E_k) \\ &= P(\text{player } k \text{ wins} \mid E_k)P(E_k) \\ &= p_0 P(E_k). \end{aligned}$$

The final equality holds because once the bill arrives at position k, the player at position k has the same likelihood of winning the game as the player at position 0 did initially. It follows from Theorem 3 that for $k > 0$

$$P(E_k) = \lambda^k.$$

Walks on a Lattice

This result lends itself quite naturally to the following setting: Imagine a random walk on the two-dimensional integer lattice. A walker begins at the origin, and at each step he moves one unit left, one unit right, or one unit up, each with probability $1/3$. (Once the walker moves up, he can never move back down.) Let k be a nonzero integer. Suppose that the walk will terminate if and only if the walker arrives at position $(k, 0)$. What is the probability that such a walk will terminate?

The answer is $P(E_k) = \lambda^{|k|}$.

A fundamental question regarding random walks on an integer lattice is the following: What is the probability that the walker returns to his initial position? This question was posed by Pólya for walks on the one-dimensional lattice (see, for example, [**4**]), and it is easily answered in our setting. Let E_0 be the event that after a *positive* number of moves, the walker returns to position 0, and denote by $\overline{E}_0$ its complement: the event that the walker never returns to position 0. Then

$$\begin{aligned} \frac{1}{\sqrt{5}} &= P(\text{Player 0 wins}) \\ &= P(\text{Player 0 wins } \cap\ E_0) + P(\text{Player 0 wins } \cap\ \overline{E}_0) \\ &= P(\text{Player 0 wins} \mid E_0)P(E_0) + P(\text{Player 0 wins } \cap\ \overline{E}_0) \\ &= \frac{1}{\sqrt{5}}P(E_0) + \frac{1}{3}, \end{aligned}$$

from which it follows that $P(E_0) = \sqrt{5}(\frac{1}{\sqrt{5}} - \frac{1}{3}) = \frac{3-\sqrt{5}}{3} \approx 0.25464$.

We note that the arguments presented thus far can be modified to make use of properties of the Catalan numbers. Alternately, using these results we can deduce properties of the Catalan numbers. For example: Recall that the nth Catalan number C_n counts the number of sequences comprising n Ls and n Rs, and such that the number of Rs never exceeds the number of Ls among the first k terms, for $1 \leq k \leq 2n$. The event E_1 occurs precisely if there is an n for which the initial sequence of left and right moves has precisely these properties—so the bill returns to the origin after $2n$ moves without ever moving to the right of the origin—and then we append a final R to the sequence to represent the last move, from position 0 to position 1. So E_1 has probability

$$\sum_{n=0}^{\infty} \frac{C_n}{3^{2n+1}}.$$

Since $P(E_1) = \lambda = (3 - \sqrt{5})/2$, it follows immediately that this series converges to λ.

We note also that in the general setting of our left-right-up random walk on an integer lattice, a natural problem is to find the probability that a walk will arrive at the lattice point (x, y). Using lattice Green functions, it is possible to answer this question completely. See, for example, [**3**] and [**6**].

The Case of Finitely Many Players, Revisited

Theorem 2 gave an expression for $P_{n,k}$ in terms of Fibonacci and Lucas numbers. We now give a closed-form expression for this probability.

Theorem 4. $P_{n,k} = p_0 \frac{\lambda^k + \lambda^{n-k}}{1-\lambda^n}$, *where* $p_0 = \frac{1}{\sqrt{5}}$, *and* $\lambda = \frac{3-\sqrt{5}}{2}$.

Proof. In order for the player in position k to win, there must first be a sequence of bill movements that result in the bill landing at position k. Consider the entire sequence S of left and right moves that the bill makes, right up to the move before the winning move is made from position k. We can encode this sequence as an ordered list of Ls and Rs, where the leftmost entry denotes the first move made by the bill and the rightmost entry the last. If exactly k Ls are appended to this list, it represents a path starting and ending at position 0. This path has a *winding number* about the center of the table—the number of complete counterclockwise revolutions minus the number of complete clockwise revolutions. See, for example, [**1**]. If the winding number for this path is q, and if the sequence S has m_L left moves and m_R right moves, then

$$m_R - m_L = qn + k.$$

The key observation is this: The probability that the player in position k wins via a move-sequence with winding number q is precisely p_{qn+k}, the probability that the player in position $qn + k$ wins in a game with infinitely many players. For in both cases the bill is moving a net $qn + k$ positions before the winning move is made. So we have

$$\begin{aligned} P_{n,k} &= \sum_{q \in \mathbb{Z}} P((\text{player } k \text{ wins}) \cap (m_R - m_L = qn + k)) \\ &= \sum_{q \in \mathbb{Z}} p_{qn+k} \\ &= \sum_{q \in \mathbb{Z}, q \geq 0} p_{qn+k} + \sum_{q \in \mathbb{Z}, q < 0} p_{qn+k} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{q\in\mathbb{Z},q\geq 0} p_{qn+k} + \sum_{q\in\mathbb{Z},q>0} p_{-qn+k} \\
&= \sum_{q\in\mathbb{Z},q\geq 0} p_{qn+k} + \sum_{q\in\mathbb{Z},q>0} p_{qn-k} \qquad (\text{since } p_j = p_{-j}) \\
&= \sum_{q\in\mathbb{Z},q\geq 0} p_{qn+k} + \sum_{q\in\mathbb{Z},q\geq 0} p_{qn+(n-k)}.
\end{aligned}
$$

Letting $\lambda = (3-\sqrt{5})/2$, and noting that for $j \geq 0$, $p_j = p_0\lambda^j$ by Theorem 3, we have

$$
\begin{aligned}
P_{n,k} &= p_0\left(\sum_{q=0}^{\infty}\lambda^{qn+k} + \sum_{q=0}^{\infty}\lambda^{qn+(n-k)}\right) \\
&= p_0\left(\lambda^k\sum_{q=0}^{\infty}(\lambda^n)^q + \lambda^{n-k}\sum_{q=0}^{\infty}(\lambda^n)^q\right) \\
&= p_0\left(\lambda^k + \lambda^{n-k}\right)\sum_{q=0}^{\infty}(\lambda^n)^q \\
&= p_0\frac{\lambda^k + \lambda^{n-k}}{1-\lambda^n},
\end{aligned}
$$

since $0 < \lambda^n < 1$.

We return now to our opening question: If you are one of n people seated around a table playing this game of chance, and the bill is in front of you, how likely are you to win the game? If it is just you, Leonardo, and Édouard, your chance of winning is $F_3/L_3 = 2/4 = 1/2$. It seems intuitively clear that as the number of players increases, your chance of winning decreases. Indeed, Theorem 4 provides a simple means for rigorously establishing this fact. When there are infinitely many players, we've established that your chance of winning is $1/\sqrt{5} \approx 0.4472$. Therefore, in a game with n players your chance of winning is at most $1/2$ (when there are three players), and as the number of players increases, it decreases toward the limiting value of $1/\sqrt{5}$. So while your winning probability falls within this narrow range, affected only slightly by the number of players in the game, it is delightful to understand that your (and every other player's) winning probability can be expressed as a simple ratio with Fibonacci and Lucas numbers.

More importantly, this game of chance provides perhaps the simplest means for understanding the ratio F_n/L_n as a *probability*. While we would be delighted to be shown otherwise, we have not found any similar examples in the literature.

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Summary. A simple game of chance is introduced, where it is shown that each player's winning probability is a ratio of Fibonacci and Lucas numbers.

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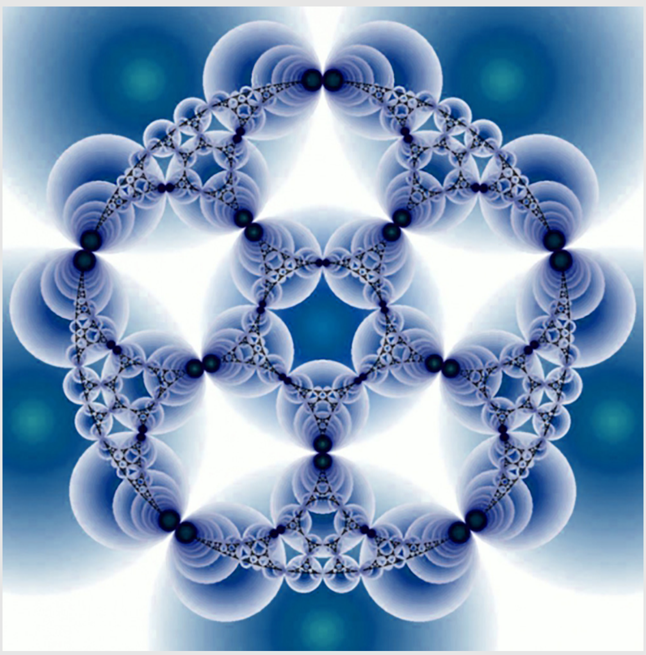

Artist Spotlight
Anne Burns

Inversions Five, Anne Burns; digital print, 2012. Five pairwise tangent circles are all tangent to a sixth circle centered at the origin. The discs bounded by these six circles are colored in blue-green. An iterated function system is made up of repeated inversions in the six circles.

See interview on page 375.

Characterizations of Quadratic Polynomials

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On New Year's Eve, 2014, I was pleasantly surprised to find a copy of issue No. 6 of Vol. 33 of MAA Focus in my letter box. Of its feature articles, I found the article by Jeffrey Forrester, Jennifer Schaefer, and Barry Tesman, [**5**], titled "But my Physics teacher said..." particularly appealing. Seemingly, this was provoked by an interaction between one of its authors, a teacher of an introductory course of single-variable calculus, and one of her students, who was also taking a physics course. Apparently, the latter had obtained the correct answer to an examination question about the average velocity of a particle over a time interval by using a recipe he claimed to have learned in his physics course. According to this, by his account, the average velocity is equal to the arithmetic mean of the instantaneous velocities at the ends of the interval. As was pointed out by the authors of [**5**], this is certainly true if the particle moves with constant acceleration—which just happened to be the case described in the question—but that it's not generally true. The main purpose of this note is to prove that the first part of this statement holds *only* for particles that have constant acceleration. In geometric terms, it is proved that if the slope of the line joining any two distinct points on the graph of a smooth function is equal to the arithmetic mean of its slopes at the points, then the graph is either that of a linear function or a parabola. In effect, this means that the identity that the student implicitly used in his answer provides an intrinsic characterization of quadratic polynomials. This is demonstrated in the next section. Our discussion there suggests another condition that turns out also to be characteristic of quadratic polynomials; this is examined in the succeeding section. The last section deals with a feature of the mean value theorem of calculus that is exhibited only by quadratic polynomials.

The student's assumption

By the average velocity of a moving object along a straight line is meant as the ratio of the distance travelled (the net displacement) over the time taken. In other words, if the object has position $f(t)$ at time t, its average velocity between consecutive times $t = x$ and $t = y$ is measured by

$$\frac{f(y) - f(x)}{y - x}.$$

The aforementioned student appeared to assume that this is always equal to the arithmetic mean of the instantaneous velocities at $t = x$ and $t = y$, i.e., that

$$\frac{f(y) - f(x)}{y - x} = \frac{f'(y) + f'(x)}{2}, \quad \forall x, y \in \mathbb{R}, x \neq y,$$

and got the correct answer, because, fortunately for him, the object moved with constant acceleration.

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MSC: Primary 39B22

Indeed, a simple computation shows that this holds if f is any quadratic polynomial, real or complex; and it's been known for sometime that this relation holds only for such functions. For instance, Cargo [**2**] obtained this conclusion in 1977 by using simple calculus methods. In fact, Cargo showed a bit more, namely, that if f is differentiable on the real line $\mathbb{R}$, and any two of the following expressions

$$\frac{f(y)-f(x)}{y-x}, \quad \frac{f'(y)+f'(x)}{2}, \quad f'\left(\frac{x+y}{2}\right)$$

are equal for all pairs of distinct real numbers (x, y), then f is a quadratic polynomial. Later on, Haruki [**6**] and Aczel [**1**] conjured up more general means by replacing f' in the last display by an arbitrary (measurable) function g, and achieved the same result without using calculus methods. Guided by their arguments, we present an algebraic proof of the observation that sparked our interest.

Theorem 1. *Suppose f is differentiable on the real line $\mathbb{R}$. Then*

$$\frac{f(y)-f(x)}{y-x} = \frac{f'(y)+f'(x)}{2}, \quad \forall x, y \in \mathbb{R}, x \neq y \tag{1}$$

holds if, and only if, f is a quadratic polynomial.

Proof. Suppose f is a quadratic polynomial. Then it is a linear combination of the monomials 1, x, and x^2, each of which is easily seen to satisfy equation (1). Hence, by linearity, so does f.

Conversely, assume f is differentiable on $\mathbb{R}$ and satisfies equation (1). By setting $y = 0$ in equation (1), we see first that

$$f(x) = f(0) + x\left(\frac{f'(x)+f'(0)}{2}\right), \quad \forall x \in \mathbb{R}, \tag{2}$$

and as a consequence that

$$y(f'(y)+f'(0)) - x(f'(x)+f'(0)) = (y-x)(f'(y)+f'(x)), \quad \forall x, y \in \mathbb{R}.$$

By expanding this, and rearranging the resulting expression, we find that

$$y(f'(x)-f'(0)) = x(f'(y)-f'(0)), \quad \forall x, y \in \mathbb{R}. \tag{3}$$

Choose a nonzero y and put

$$a = \frac{f'(y)-f'(0)}{2y}.$$

Then, by equation (3), $f'(x) = f'(0) + 2ax,\ \forall x \in \mathbb{R}$, and feeding this back into equation (2), we see that

$$f(x) = f(0) + x\left(\frac{f'(x)+f'(0)}{2}\right) = f(0) + x(f'(0)+ax) = f(0) + f'(0)x + ax^2.$$

Thus, f is a polynomial of degree at most 2. ∎

Remark. If, more generally, we consider the motion of a particle in 3-space, whose position vector is given by $\mathbf{r(t)} = (\mathbf{x_1(t)}, \mathbf{x_2(t)}, \mathbf{x_3(t)})$, we're led to look for solutions of the equation

$$\frac{\mathbf{r(s)}-\mathbf{r(t)}}{s-t} = \frac{\mathbf{r'(s)}+\mathbf{r'(t)}}{2}, \quad \forall s, t \in \mathbb{R}, s \neq t.$$

But nothing new emerges, since this is equivalent to the following three scalar equations:

$$\frac{x_i(s)-x_i(t)}{s-t}=\frac{x_i'(s)+x_i'(t)}{2}, \quad i=1,2,3, \quad \forall s,t\in\mathbb{R}, s\neq t.$$

Remark. If, on the other hand, we substitute "speed" for "velocity" in the description of the student's assumption, the character of the problem changes, and it becomes: For what differentiable functions f on $\mathbb{R}$ is it true that, if $x, y\in\mathbb{R}$, and $x\neq y$, then

$$\frac{1}{y-x}\int_x^y |f'(t)|\,dt=\frac{|f'(y)|+|f'(x)|}{2}?$$

In the first place, for this to be meaningful, it must be assumed that $|f'|$ is locally integrable in some sense. Interpreting the integral in the Lebesgue sense, the reader is invited to confirm that the only nonnegative locally Lebesgue integrable functions g that satisfy the equation

$$\frac{1}{y-x}\int_x^y g(t)\,dt=\frac{g(x)+g(y)}{2}, \quad \forall x,y\in\mathbb{R}, x\neq y$$

are constants.

A consequential problem

For completeness, we present a description of the solution set of the differential equation (2), which arises naturally in the previous discussion. Of course, every quadratic polynomial satisfies this equation, but the converse is false: the differentiable function $x\to x|x|$ is a simple counterexample.

Theorem 2. *Suppose f is defined on $\mathbb{R}$. Then f is differentiable on $\mathbb{R}$, and satisfies equation (2) if, and only if, there are quadratic polynomials p, q, whose derivatives agree at* 0*, such that $f=p$ on* $(-\infty,0]$*, and $f=q$ on* $[0,\infty)$.

Proof. Suppose f is differentiable, and satisfies equation (2). By definition of the derivative at 0,

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x},$$

and so, by equation (2),

$$\frac{f'(0)+\lim_{x\to 0}f'(x)}{2}=f'(0).$$

Hence,

$$f'(0)=\lim_{x\to 0}f'(x).$$

Thus, f' is continuous at 0. But in any event, $(f(x)-f(0))/x$ is differentiable on $\mathbb{R}\setminus\{0\}$. Hence, equation (2) tells us that f' is differentiable on $\mathbb{R}\setminus\{0\}$, and, since

$$\frac{f'(x)+f'(0)}{2}=\frac{f(x)-f(0)}{x}, \quad x\neq 0,$$

an application of the product rule for differentiation shows that

$$\frac{1}{2}f''(x) = \frac{f'(x}{x} - \frac{f(x) - f(0)}{x^2} = \frac{f'(x)}{x} - \frac{f'(x) + f'(0)}{2x}, \quad x \neq 0,$$

we see that

$$\frac{f'(x) - f'(0)}{x} = f''(x), \quad \forall x \in \mathbb{R}, x \neq 0.$$

A repeat argument shows that f'' is also differentiable on $\mathbb{R} \setminus \{0\}$, and that

$$f'''(x) = \frac{xf''(x) - (f'(x) - f'(0))}{x^2} = \frac{xf''(x) - xf''(x)}{x^2} = 0, \quad \forall x \in \mathbb{R}, x \neq 0.$$

Thus, f'' is identically zero on each of the intervals $(-\infty, 0)$, $(0, \infty)$. Therefore, there are two quadratic polynomials p, q such that $f(x) = p(x)$, $\forall x \leq 0$, and $f(x) = q(x)$, $\forall \geq 0$. It follows that $f'(x) = p'(x)$, $\forall x < 0$, and $f'(x) = q'(x)$, $\forall x > 0$. Hence,

$$f'(0^-) = \lim_{x \to 0^-} f'(x) = p'(0), \quad \text{and } f'(0^+) = \lim_{x \to 0^+} f'(x) = q'(0),$$

and so, since f' is continuous at 0, $p'(0) = q'(0)$. This establishes the necessity of the conditions.

Conversely, if p, q are quadratic polynomials such that $p'(0) = q'(0)$, and $f(x) = p(x)$, $\forall x \leq 0$, and $f(x) = q(x)$, $\forall x \geq 0$, then, in the first instance, f is differentiable on $\mathbb{R} \setminus \{0\}$. Moreover,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{p(x) - p(0)}{x} = p'(0)$$

and

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{q(x) - q(0)}{x} = q'(0).$$

Since $p'(0) = q'(0)$, f is also differentiable at 0, and $f'(0) = p'(0) = q'(0)$. But p, q satisfy equation (2). Hence, if $x \leq 0$,

$$f(x) - f(0) - x\left(\frac{f'(x) + f'(0)}{2}\right) = p(x) - p(0) - x\left(\frac{p'(x) + p'(0)}{2}\right) = 0.$$

Thus, f satisfies equation (2) on $(-\infty, 0]$. Similarly, it satisfies this equation on $[0, \infty)$. This establishes the sufficiency of the conditions. ■

It should be apparent that one can use this result to prove Theorem 1. Indeed, equation (1) implies equation (2), and so there are four numbers a_-, a_+, b, and c such that

$$f(x) = \begin{cases} a_- x^2 + bx + c, & \text{for } x \leq 0, \\ a_+ x^2 + bx + c, & \text{for } x \geq 0, \end{cases}$$

in which case, by equation (1), if $x > 0$,

$$\begin{aligned} (a_+ - a_-)x + 2b &= \frac{f(x) - f(-x)}{x} \\ &= f'(x) + f'(-x) \\ &= 2(a_+ - a_-)x + 2b. \end{aligned}$$

Thus, $a_+ = a_-$, and f is a quadratic polynomial.

A feature of the mean value theorem

According to the mean value theorem, if f is a real-valued differentiable function on $\mathbb{R}$, and x and y are distinct real numbers, there is a point z between them, such that

$$\frac{f(y) - f(x)}{y - x} = f'(z).$$

Moreover, z can be written in the form $z = (1 - \alpha)y + \alpha x$, where $\alpha \in (0, 1)$. It's easy to verify that $2\alpha = 1$ if f is a quadratic polynomial. The converse of this statement appears to have been first proved in [**7**] under the assumption that f is thrice differentiable, a condition that was shown in [**6**] to be redundant. This suggests the next result which was first examined in [**3**] under the hypothesis that f is many-times differentiable. We examine the same issue without the latter assumption, and avoid calculus in our proof. Several years after the appearance of [**3**], a slightly more general form of this functional equation was proposed by Walter Rudin as Problem E3338 in the *Amer. Math. Monthly* in 1989, a calculus solution of which appeared in the 1991 March issue of this journal. An algebraic solution of Rudin's problem is presented in the book [**8**], an excellent reference for undergraduates interested in functional equations.

Theorem 3. *Suppose* $\alpha \in (0, 1)$*, and* f *is a differentiable function on* $\mathbb{R}$ *such that*

$$\frac{f(x) - f(y)}{x - y} = f'(\alpha x + (1 - \alpha)y), \quad \forall x, y \in \mathbb{R}, x \neq y. \tag{4}$$

Then, either (i) $2\alpha \neq 1$*, in which case* f *is a polynomial of degree at most* 1*, or (ii)* $2\alpha = 1$*, in which case* f *is a polynomial of degree at most* 2.

Proof. Let $g = f'$. Then, by equation (4), $f(x) - f(0) = xg(\alpha x)$, $\forall x \in \mathbb{R}$, whence, plugging this back into equation (4) and rearranging the resulting expression,

$$yg(\alpha y) - xg(\alpha x) = (y - x)g(\alpha x + (1 - \alpha)y), \quad \forall x, y \in \mathbb{R}. \tag{5}$$

Next, letting $\zeta = (1 - \alpha)/\alpha$, and changing variables, we see that

$$yg(y) - xg(x) = (y - x)g(x + \zeta y), \quad \forall x, y \in \mathbb{R}. \tag{6}$$

We proceed to solve this functional equation, without imposing any additional analytical conditions on g. Clearly we may suppose that $g(0) = 0$; otherwise, we replace g by $g - g(0)$. With this assumption in place, letting $x = -\zeta y$ in equation (6), we see that

$$yg(y) - (-\zeta y)g(-\zeta y) = 0, \quad \forall y \in \mathbb{R},$$

i.e., $yg(y) = -\zeta yg(-\zeta y)$. Replacing y by $-\zeta y$ in equation (6), we deduce that

$$\begin{aligned}(y - x)g(x + \zeta y) = yg(y) - xg(x) &= -\zeta yg(-\zeta y) - xg(x)\\ &= -(x + \zeta y)g(x - \zeta^2 y).\end{aligned}$$

In other words,

$$(y - x)g(x + \zeta y) = -(x + \zeta y)g(x - \zeta^2 y), \quad \forall x, y \in \mathbb{R}. \tag{7}$$

Hence, taking $x = \zeta^2 y$, we have that

$$(1 - \zeta^2)yg(\zeta(1 + \zeta)y) = 0, \quad \forall y \in \mathbb{R}.$$

Consequently, unless $\zeta^2 = 1$, g is the zero function. In other words, if $2\alpha \neq 1$, constants are the only solutions of equation (5). Reverting to equation (4), we deduce that if $2\alpha \neq 1$, then f is a linear function. This proves part (i).

It only remains to deal with the case $2\alpha = 1$. But this constraint forces $\zeta = 1$, so that equation (7) reduces to

$$(y - x)g(x + y) = -(x + y)g(x - y), \quad \forall x, y \in \mathbb{R},$$

equivalently, after a change of variables, $yg(x) = xg(y)$, $\forall x, y \in \mathbb{R}$. Hence, g is at most a linear function, and so part (ii) follows. Thus, whatever the value of α, f is a polynomial of degree at most 2. ∎

Acknowledgment I extend my thanks to the referees for their constructive criticism.

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Summary. This note was motivated by a student's misconception that the average velocity over a time interval of a particle in rectilinear motion is the arithmetic mean of its velocities at the ends of the interval. We present an algebraic proof that this property holds only if the particle has constant acceleration, thereby recovering a well-known result. A characterization is also provided of differentiable functions on $(-\infty, \infty)$ whose restrictions to the intervals $(-\infty, 0]$ and $[0, \infty)$ are quadratic polynomials. In addition, a calculus-free proof is presented to show that a certain feature of the mean value theorem holds only for quadratic polynomials.

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Proof Without Words: Alternating Row Sums in Pascal's Triangle

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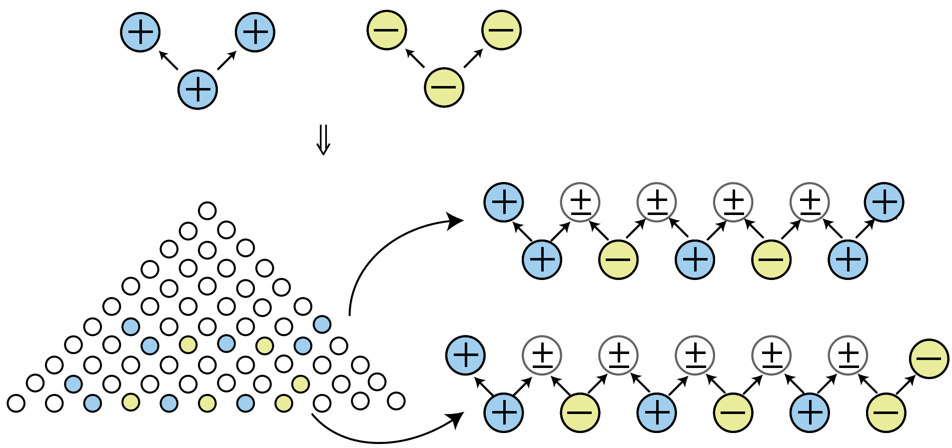
Theorem. For any integers $0 \leq j \leq m \leq n$,

$$\sum_{k=j}^{m} (-1)^k \binom{n}{k} = (-1)^j \binom{n-1}{j-1} + (-1)^m \binom{n-1}{m},$$

and in particular if $j = 0$ and $m = n$, then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ (by defining as usual $\binom{n-1}{-1} = 0 = \binom{n-1}{n}$).

Proof. For simplicity we show the case when j is even; the odd cases can be obtained by reversing the role of $+$ and $-$.



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \Rightarrow \sum_{k=j}^{m} (-1)^k \binom{n}{k} = (-1)^j \binom{n-1}{j-1} + (-1)^m \binom{n-1}{m}.$$

Summary. Based on the Pascal's identity, we visually demonstrate that the alternating sum of consecutive binomial coefficients in a row of Pascal's triangle is determined by two binomial coefficients from the previous row.

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Powers of a Class of Generating Functions

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Many well-known polynomial sequences $\mathbf{v} = \{v_n(x)\}_{n\geq 0}$ such as those of Chebyshev, Fibonacci, Lucas, Pell, Jacobsthal, Morgan-Voyce, and Fermat are generated by the rational function

$$F(x,z) = \sum_{n\geq 0} v_n\, z^n = \frac{a(x)+b(x)z}{1-p(x)\,z+q(x)\,z^2}, \qquad q \neq 0, \tag{1}$$

for appropriate choices of polynomials $p(x)$, $q(x)$, $a(x)$, and $b(x)$. For instance, with $p = 2x$, $q = 1$, $a = 1$ and $b = -x$ or 0, we have the well-known functions $T(x,z)$ and $U(x,z)$ that generate the Chebyshev polynomials of the first and second kind, respectively, and with $x = 3$ in the latter, we obtain the sequence of numbers a_n whose squares are triangular numbers. Thus,

$$\begin{aligned} U(3,z) &= \frac{1}{1-6z+z^2} = \sum_{n\geq 0} a_n\, z^n \\ &= 1+6z+35z^2+204z^3+1189z^4+6930z^5+\cdots. \end{aligned}$$

Motivated by the fact that the square root of $U(x,z)$ is a generating function for the Legendre polynomials (described in Example 1 below), we consider the series expansion of $F(x,z)^\alpha$, where α is a positive number, and the sequences generated by these functions. The coefficients of $F(x,z)^\alpha$ can actually be expressed through the Faà di Bruno formula as described by Flanders [**5**]. However, as he points out, implementing this formula is inefficient as well as computationally challenging. We show that we can forgo the Faà formula and derive a recurrence for these coefficients that is more user-friendly.

Although the ordinary generating function $F(x,z)$ may depend on a parameter x as well as the variable z, we focus our attention on the latter viewpoint and write $F(z)$, suppressing the variable x. Indeed, $F(z)$ is a rational function of z, and the sequence of coefficients v_n of its Maclaurin expansion satisfies the three-term recurrence

$$v_{n+1} = p\, v_n - q\, v_{n-1}, \qquad v_0 = a, \quad v_1 = b + ap. \tag{2}$$

This recurrence is present in the denominator of the function $F(z)$. Likewise, if k is a positive integer, then the sequence of coefficients of the Maclaurin expansion of $F(z)^k$ satisfies a multiterm recurrence with constant coefficients as given by the expression $(1 - p(x)\,z + q(x)\,z^2)^k$. In contrast, we will derive a four-term recurrence for the sequence of coefficients generated by $F(z)^\alpha$ for any positive number α.

Let us expand $F(z)^\alpha$ into a Maclaurin series:

$$F(z)^\alpha = \left(\sum_{n\geq 0} u_n\, z^n\right)^\alpha = \sum_{n\geq 0} r_n\, z^n = \left(\frac{a(x)+b(x)z}{1-p(x)\,z+q(x)\,z^2}\right)^\alpha, \tag{3}$$

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where the sequence of coefficients $\mathbf{r} = \{r_n\}_{n\geq 0}$, which depends on α, is to be determined. We write $F_\alpha(z)$ for $F(z)^\alpha$. At this point, we have not placed any constraint on the positive number α. When $\alpha = 2$, series (3) generates the convolution of the sequence $\mathbf{u} = \{u_n\}_{n\geq 0}$ with itself:

$$\mathbf{r} = \mathbf{u} \star \mathbf{u} \qquad \text{or} \qquad r_n = \sum_{j=0}^{n} u_j\, u_{n-j}.$$

More generally, when α is a positive integer, $F_\alpha(z)$ generates the sequence $\mathbf{r} = \{r_n\}_{n\geq 0}$; it is called the αth convolution of $\mathbf{u}$ and is denoted

$$\mathbf{r} = \underbrace{\mathbf{u} \star \mathbf{u} \star \cdots \star \mathbf{u}}_{\alpha \text{ times}}.$$

Finding explicit expressions for convoluted sequences has attracted much attention (see, for example, [**1, 3, 5, 6, 7, 9, 10, 11, 13**]). A more difficult and less studied problem occurs when α is a rational number. Of particular importance is the case where α is the reciprocal of a positive integer m. When $\alpha = 1/m$, the function $F_{1/m}(z) = \left(\sum_{n\geq 0} u_n\, z^n\right)^{1/m} = \sum_{n\geq 0} r_n\, z^n$ generates the sequence $\mathbf{r} = \{r_n\}$, where $\mathbf{u} = \underbrace{\mathbf{r} \star \mathbf{r} \star \cdots \star \mathbf{r}}_{m \text{ times}}$ is a known sequence. Here we are dealing with the positive mth root. In this case, it is natural to call $\mathbf{r} = \{r_n\}_{n\geq 0}$ the mth root sequence of the sequence $\mathbf{u}$. For example, if $a = 1$, $b = 0$ and $\alpha = 1/4$,

$$\begin{aligned} \frac{1}{(1 - pz + qz^2)^{1/4}} &= \sum_{n\geq 0} r_n\, z^n \\ &= 1 + \frac{pz}{4} + \frac{z^2}{32}\,(5p^2 - 8q) + \frac{5pz^3}{128}\left(3p^2 - 8q\right) + \cdots \end{aligned}$$

yields a fourth root sequence. When $\alpha = k/m$ is the ratio of two positive relatively prime integers k and m, the function $F_{k/m}(z)$ generates the sequence that is the kth convolution of the mth root of the given sequence $\mathbf{u}$. At first glance, this problem for $\alpha = k/m$ seems to be intractable. Finding the coefficients of the generating function $F_\alpha(z)$ using the Faà di Bruno formula is not practical because it involves summation based on the partitions of the index, and the partitioning of a positive integer grows exponentially [**12**].

Since the given sequence $\mathbf{u}$ is a solution of the homogeneous constant-coefficient difference equation (2), it is possible to show that the sequence $\mathbf{r}$ is also a solution of a three-term recurrence but with variable coefficients. To simplify computations, we write the generating function (1) as a product of two simple factors, $F(z) = (a + bz)\, G(z)$, and work with

$$G_\alpha(z) = \sum_{n\geq 0} r_n\, z^n = \frac{1}{(1 - p(x)\, z + q(x)\, z^2)^\alpha}. \tag{4}$$

In order to derive this recurrence for coefficients r_n, we differentiate $G(z)$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\, G_\alpha(z) = \sum_{n\geq 1} r_n\, n\, z^{n-1} = \sum_{n\geq 0} r_{n+1}\,(n+1)\, z^n.$$

On the other hand, its derivative from the right-hand side of (4) is

$$\frac{\mathrm{d}}{\mathrm{d}z}\, G_\alpha(z) = \alpha\,(p - 2qz)\, \frac{G_\alpha(z)}{1 - pz + qz^2}.$$

We equate both expressions for the derivative and multiply through by $1 - p\,z + q\,z^2$, obtaining

$$(1 - pz + qz^2) \sum_{n\geq 0} r_{n+1}\,(n+1)\,z^n = \alpha(p - 2qz) \sum_{n\geq 0} r_n\,z^n.$$

Adjusting indices appropriately and extracting coefficients of z^n, we arrive at the following three-term recurrence that generalizes (2) when $a = 1$ and $b = 0$:

$$(n+1)\,r_{n+1} = p\,(\alpha + n)\,r_n - q\,(2\alpha + n - 1)\,r_{n-1}, \quad n \geq 1, \tag{5}$$

where $r_0 = 1,\ r_1 = \alpha p$. When the generating function $F(z) = (a + bz)G(z)$ is raised to the power α, it can be shown in a similar way that the coefficients of its Maclaurin expansion satisfy the four-term recurrence

$$\begin{aligned} a\,(n+1)\,r_{n+1} &= (ap(\alpha+n) + b(\alpha - n))\,r_n - (aq(2\alpha + n - 1) - bp(n-1))\,r_{n-1} \\ &\quad - bq\,(\alpha + n - 2)\,r_{n-2}, \qquad n = 2, 3, \ldots, \end{aligned} \tag{6}$$

where $r_0 = a^\alpha$, $r_1 = \alpha a^{\alpha-1}(b + ap)$, and $r_2 = (1/2)a^{\alpha-2}\alpha(b^2(\alpha - 1) + 2ab\alpha p + a^2((\alpha+1)p^2 - 2q))$. This recurrence reduces to (5) when $a = 1$ and $b = 0$.

We close our exposition with four illustrative examples.

Example 1. The Legendre polynomials $P_n(x)$ and Chebyshev polynomials have found countless applications in all branches of engineering and science. The generating functions for the sequences of Legendre polynomials and Chebyshev polynomials $U_n(x)$ of the second kind are well known [**4**]:

$$U(x, z) = \sum_{n\geq 0} U_n(x)\,z^n = \frac{1}{1 - 2xz + z^2},$$

$$P(x, z) = \sum_{n\geq 0} P_n(x) z^n = \frac{1}{(1 - 2xz + z^2)^{1/2}}.$$

The sequence of Legendre's polynomials is the "square-root sequence" of the sequence of Chebyshev polynomials; the sequence $\{P_n(x)\}$ satisfies a familiar recurrence (5), with $p = 2x$, $q = 1$, and $\alpha = 1/2$.

The first few Chebyshev and Legendre polynomials are

$$U_0 = 1,\ U_1 = 2x,\ U_2 = 4x^2 - 1,\ U_3 = 8x^3 - 4x,\ U_4 = 16x^4 - 12x^2 + 1,\ \ldots,$$

$$P_0 = 1,\ P_1 = x,\ P_2 = \frac{1}{2}\left(3x^2 - 1\right),\ P_3. = \frac{1}{2}\left(5x^3 - 3x\right), \ldots.$$

Example 2. Consider the well-known [**2, 8**] sequence $\{a_n\}$ of numbers whose squares are triangular numbers. That is, a_n satisfies the Diophantine equation $a_n^2 = M(M + 1)/2$ for some integer M. There is an explicit formula for such numbers using the floor function:

$$a_n = \frac{(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}}{4\sqrt{2}} = \left\lfloor \frac{(3 + 2\sqrt{2})^{n+1}}{4\sqrt{2}} \right\rfloor, \quad n = 0, 1, 2, \ldots.$$

This is actually $U_n(x)$ evaluated at $x = 3$ from Example 1. Thus, the generating function for the sequence $\{a_n\}$ becomes

$$A(z) = \frac{1}{1 - 6z + z^2} = \sum_{n\geq 0} a_n\,z^n,$$

and its square root is the Legendre function $P(3, z)$, namely

$$D(z) = \sum_{n\geq 0} d_n\, z^n = \frac{1}{\sqrt{1-6z+z^2}} = 1 + 3z + 13\, z^2 + 63\, z^3 + 321\, z^4 + 1683\, z^5 + \cdots .$$

This function generates the sequence $\{d_n\}_{n\geq 0}$ of (central) Delannoy numbers, which can be defined as the solution of recurrence (5) with $p = 6$, $q = 1$, and $\alpha = 1/2$:

$$(n+1)\, d_{n+1} = 3(2n+1)\, d_n - n\, d_{n-1}, \qquad d_0 = 1,\ d_1 = 3,\ d_2 = 13.$$

The central Delannoy number d_n is the number of paths from the southwest corner (0, 0) of an integer grid to the northeast corner (n, n), using only single steps from one vertex to the next going north, northeast, or east (i.e., steps in the direction (1,0), (1,1), and (0,1)).

Example 3. While explicit expressions for Chebyshev and Legendre polynomials are well known [**4**], the square root of the latter has not been exposed:

$$\begin{aligned} U^{1/4}(x, z) = P^{1/2}(x, z) &= \frac{1}{(1 - 2xz + z^2)^{1/4}} \\ &= 1 + \frac{xz}{2} + \frac{z^2}{8}\left(5x^2 - 2\right) + \frac{5xz^3}{16}\left(3x^2 - 2\right) + \frac{5z^4}{128}\left(4 - 36x^2 + 39x^4\right) + \cdots . \end{aligned}$$

This generating function gives birth to the sequence of polynomials that are related by recurrence (5) with $p = 2x$, $q = 1$, and $\alpha = 1/4$.

Example 1 naturally leads one to wonder about the square root sequence for the Chebyshev polynomials $T_n(x)$ of the first kind.

Example 4. The generating function for polynomials $T_n(x)$ is

$$T(x, z) = \sum_{n\geq 0} T_n(x)\, z^n = \frac{1 - xz}{1 - 2xz + z^2}.$$

Its square root sequence is given by

$$\begin{aligned} T^{\frac{1}{2}}(x, z) = \sum_{n\geq 0} r_n(x)\, z^n &= \frac{(1 - xz)^{\frac{1}{2}}}{(1 - 2xz + z^2)^{\frac{1}{2}}} \\ &= 1 + \frac{x}{2}z + \frac{7x^2 - 4}{2^3}z^2 + \frac{25x^3 - 20x}{2^4}z^3 + \frac{363x^4 - 376x^2 + 48}{2^7}z^4 + \cdots . \end{aligned}$$

The polynomials $r_n(x)$ satisfy the four-term recurrence (6) with $\alpha = 1/2$:

$$(n+1)\, r_{n+1} = x(3n + 1/2)r_n - (2x^2(n-1) + n)r_{n-1} + x(n - 3/2)r_{n-2}, \quad n \geq 1.$$

The reader may wish to experiment with the generating functions of some of the polynomial sequences mentioned at the outset.

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Summary. We consider positive powers of a rational generating function $F(x, z)$ of the variable z whose numerator has degree at most 1, denominator has degree 2, and where coefficients are functions of x. Many well-known polynomial sequences are generated by functions of this form, including the generating function $U(x, z)$ for Chebyshev polynomials $U_n(x)$ of the second kind; the square root $U(x, z)^{1/2}$ generates the sequence of Legendre polynomials. The numerical sequence $U_n(3)$ is the sequence of numbers whose squares are triangular numbers, and the square root of its generating function breeds the sequence of central Delannoy numbers. Our goal is to provide a recurrence for the sequence generated by $F(x, z)^\alpha$, where α is a positive real number, thus providing a reasonable way for computing sequential values.

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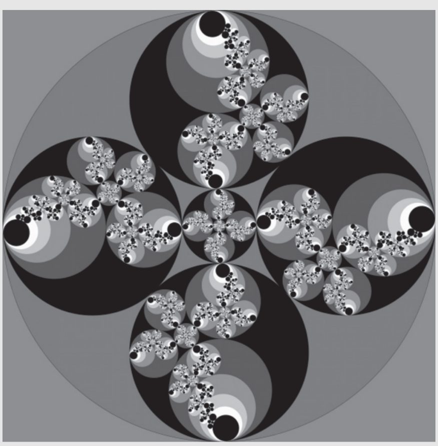

Artist Spotlight
Anne Burns

Tangent Circles I, Anne Burns; digital print, 2015. An iterated function system is comprised of five complex functions acting on five circles; four of the circles are pairwise tangent and tangent to the unit circle and the fifth circle is centered at the origin and tangent to the other four.

See interview on page 375.

A Bounded Derivative That Is Not Riemann Integrable

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An important goal for those of us who teach elementary real analysis is to help students view broader mathematical vistas. Presenting examples of functions that exhibit strange behavior is one way to achieve this goal. For instance, the existence of continuous functions that are nowhere differentiable can be quite a surprise for students. As the title of this note indicates, we are interested in differentiable functions whose derivatives are not Riemann integrable. The existence of such functions explains the need for a seemingly extra hypothesis in the statement of the fundamental theorem of calculus and illustrates that a more powerful integration process is necessary for higher level analysis.

The fundamental theorem of calculus

In simple, but not precise, language, the fundamental theorem of calculus states that differentiation and integration are inverse processes. Since the two operations can occur in either order, there are two parts of the theorem. For our purposes, we are interested in the following statement.

- If the function f is Riemann integrable on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f = F(b) - F(a)$.

By incorporating the definition of antiderivative into the hypothesis, we obtain

- If F is differentiable on $[a, b]$ and the function F' is Riemann integrable on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$.

The second part of the hypothesis seems redundant. Since integration is (informally) the inverse operation of differentiation, aren't all derivatives integrable? (The reader may wish to consult [**4**] for an insightful discussion of the history behind the statement of the fundamental theorem of calculus.) However, once we recall that Riemann integrable functions must be bounded, an example of a derivative that is not Riemann integrable is close at hand. For example, the derivative of the function F defined by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$ exists at all points, but the function F' is not bounded on [0, 1]. (For the record, the function F is not absolutely continuous on [0, 1], so F' is not even Lebesgue integrable on [0, 1].) After seeing examples such as this, we might hope that at least the following statement is valid.

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- If F is differentiable on $[a, b]$ and F' is bounded on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$.

Unfortunately, this version of the fundamental theorem of calculus is not valid for the Riemann integral; there is no guarantee that bounded derivatives are Riemann integrable. (It does follow, however, that all bounded derivatives are Lebesgue integrable.) As we might expect, creating such an unusual function is rather involved. However, with some care, the ideas are accessible to students in an elementary real analysis course.

Brief historical background

As a start, consider the function G defined by $G(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $G(0) = 0$. The function G is differentiable at every point, and G' is bounded on any interval of the form $[a, b]$. In addition, the function G' is continuous everywhere except at $x = 0$. As a result, it is Riemann integrable on [0, 1] and the fundamental theorem of calculus holds. By slightly modifying the function G and placing its analogues in the open intervals constituting the complement of a perfect nowhere dense set of positive measure, Volterra [**13**] was able to construct a bounded derivative that was not Riemann integrable. The details tend to be rather tedious, so very few elementary real analysis texts include this function. In fact, other than the text [**9**, p. 183], I did not find any texts at this level that mention the existence of such functions. The more advanced books [**10**], [**11**, p. 490], and [**12**, p. 312] mention Volterra's function, with the latter having the details as a list of exercises. The book by Bressoud [**5**] presents a lengthy and helpful historical discussion, as well as many of the details, behind Volterra's function. (A summary of a talk given by Bressoud on this topic is available at [**3**].) See also [**8**, p. 35] for another source that presents the computational details necessary to show that this function has the desired properties.

A short note by Goffman in the March 1977 issue of the *American Mathematical Monthly* (see [**7**]) presents a collection of simpler examples of bounded derivatives that are not Riemann integrable. Since these functions seem to have attracted very little notice (the only reference I found was [**6**, p. 79], and it essentially recopies Goffman's work), we have decided to give a more careful treatment here of a slight variation on Goffman's examples. Our goal is to include as many of the details as possible, keeping the perspective of a typical student in an elementary real analysis course in mind. Having said this, however, there are still a fair number of unavoidable prerequisite ideas needed.

A bounded function f that is not Riemann integrable

Let $O = \bigcup_{n=1}^{\infty} (a_n, b_n)$ be an open dense set in the interval (0, 1), where the intervals (a_n, b_n) are pairwise disjoint and the inequality $\sum_{n=1}^{\infty} (b_n - a_n) \leq \frac{1}{2}$ is satisfied. Without loss of generality, we may assume that $a_i \neq b_j$ for all positive integers i and j. An explanation of the terms in the opening sentence of this paragraph, as well as a way to create such a set, can be found in a later section. For each positive integer n, choose points c_n and d_n so that

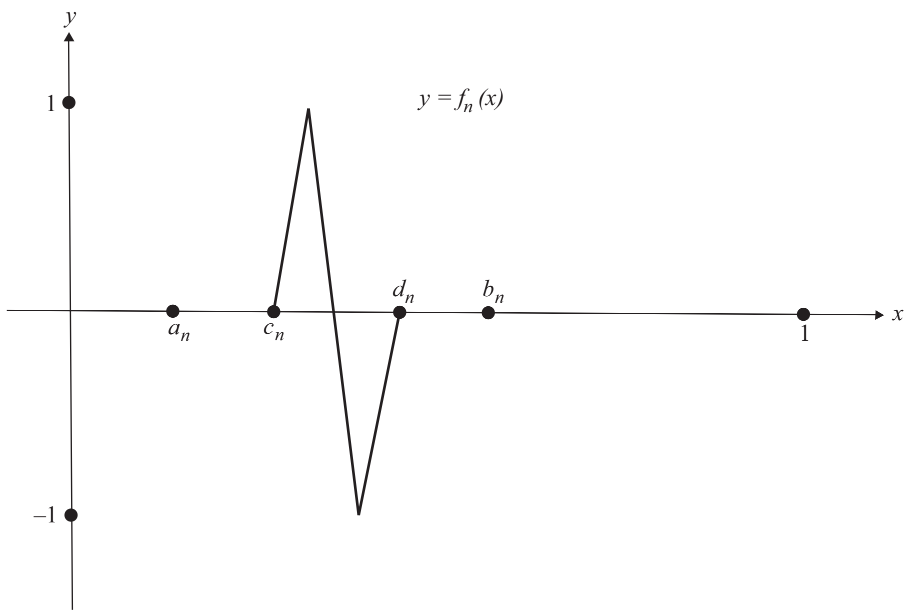
$$a_n < c_n < d_n < b_n, \quad \frac{c_n + d_n}{2} = \frac{a_n + b_n}{2}, \quad \text{and} \quad d_n - c_n = (b_n - a_n)^2.$$

In other words, the interval $[c_n, d_n]$ is a subset of (a_n, b_n) with the same center and having a length equal to the square of the length of $[a_n, b_n]$. For future reference, note that

$$c_n - a_n = \frac{(b_n - a_n) - (b_n - a_n)^2}{2} = \frac{b_n - a_n}{2}\Big(1 - (b_n - a_n)\Big) > \frac{b_n - a_n}{4},$$

where we have used the fact that $b_n - a_n < \frac{1}{2}$ for each n.

For each positive integer n, define a continuous function $f_n\colon [0, 1] \to [-1, 1]$ so that f_n is 0 on the intervals $[0, c_n]$ and $[d_n, 1]$, there is a point $t_n \in (c_n, d_n)$ such that $|f(t_n)| = 1$, and $\int_{c_n}^{d_n} f_n = 0$. There are, of course, many possibilities for f_n, but perhaps the simplest example is the piecewise linear function graphed below (the graph is not to scale).



(We note in passing that a graph of the derivative of the modified version of $x^2 \sin(1/x)$ needed for Volterra's example cannot be fully drawn on its corresponding interval so our f_n functions are more visually accessible.) Now define a bounded function $f\colon [0, 1] \to [-1, 1]$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$; the function f is well-defined since at most one term in the series is nonzero for each $x \in [0, 1]$. It is a good exercise for students to show that f is continuous at each point of the set O and that f is not continuous at each point of the set $E = [0, 1] \setminus O$. Since the set E has positive measure, the function f is not continuous almost everywhere and thus not Riemann integrable on $[0, 1]$. Since most courses in elementary real analysis do not prove that a function is Riemann integrable on $[a, b]$ if and only if it is bounded and continuous almost everywhere on $[a, b]$, we present a more elementary proof that f is not Riemann integrable on $[0, 1]$.

Since there are several ways to define the Riemann integral, we need to specify the version that we consider here. Let h be a bounded function defined on an interval $[a, b]$. The oscillation of h on $[a, b]$ is defined by

$$\omega(h, [a, b]) = \sup\{h(x) : x \in [a, b]\} - \inf\{h(x) : x \in [a, b]\}.$$

Using this definition, we obtain the following criterion for Riemann integrability.

- A bounded function h is Riemann integrable on $[a, b]$ if and only if for each positive number ϵ there exists a partition $P = \{x_i : 0 \le i \le n\}$ of $[a, b]$ such that $\sum_{i=1}^{n} \omega(h, [x_{i-1}, x_i])(x_i - x_{i-1}) < \epsilon$.

This result is either a theorem (when using Riemann sums to define the integral) or a definition (when using upper and lower sums to define the integral); see [**2**, p. 173], [**6**, p. 77], or [**9**, p. 237].

To verify that f is not Riemann integrable on [0, 1], let $\{x_i : 0 \le i \le q\}$ be a partition of [0, 1]. Define S to be the set of all indices i such that $1 \le i \le q$ and the set $(x_{i-1}, x_i) \cap E$ is empty, and let $T = \{i : 1 \le i \le q\} \setminus S$. Note that

i) if $i \in S$, then $(x_{i-1}, x_i) \subseteq O$;
ii) if $i \in T$, then $(a_n, b_n) \subseteq (x_{i-1}, x_i)$ for some n and thus $\omega(f, [x_{i-1}, x_i]) \ge 1$.

To verify observation (ii), note that the set $(x_{i-1}, x_i) \cap E$ cannot contain just one point; if so, then $a_i = b_j$ for some i and j. Suppose that $u < v$ are two points in $(x_{i-1}, x_i) \cap E$. Since O is dense in (0, 1), there is a point $t \in (u, v) \cap O$. If $t \in (a_n, b_n)$, then $(a_n, b_n) \subseteq (u, v) \subseteq (x_{i-1}, x_i)$. From these two observations and the fact that the intervals $[x_{i-1}, x_i]$ are nonoverlapping, it follows that

$$\sum_{i \in S}(x_i - x_{i-1}) \le \sum_{n=1}^{\infty}(b_n - a_n) \le \frac{1}{2} \qquad \text{and thus} \qquad \sum_{i \in T}(x_i - x_{i-1}) \ge \frac{1}{2}.$$

We then find that

$$\begin{aligned}\sum_{i=1}^{q} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) &\ge \sum_{i \in T} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) \\ &\ge \sum_{i \in T}(x_i - x_{i-1}) \ge \frac{1}{2}.\end{aligned}$$

Since this inequality holds for every partition of [0, 1], the function f is not Riemann integrable on [0, 1].

Our function f is a derivative

Now define a function F on [0, 1] by $F(x) = \sum_{k=1}^{\infty} \int_0^x f_k$. (Note once again that at most one term of this series is nonzero for each value of x.) We claim that F is differentiable on [0, 1] and that $F'(x) = f(x)$ for all $x \in [0, 1]$. For each $x \in (a_n, b_n)$, we find that $F(x) - F(a_n) = \int_{a_n}^x f_n$. Since f_n is a continuous function, the fundamental theorem of calculus (the other version!) reveals that $F'(x) = f_n(x) = f(x)$ for all $x \in (a_n, b_n)$. We conclude that $F'(x) = f(x)$ for all $x \in O$. We now prove that

$$\lim_{y \to x^+} \frac{F(y) - F(x)}{y - x} = 0$$

for each $x \in E \setminus \{1\}$. If $x = a_k$ for some positive integer k, then the result is trivial since F is 0 on the interval $[a_k, c_k]$. Assume that $x \ne a_k$ for all k and fix a value of y so that $x < y < 1$. There are two cases to consider.

I. Suppose that $y \notin \bigcup_{k=1}^{\infty}(c_k, d_k)$. Then $F(y) - F(x) = 0 - 0 = 0$.

II. Suppose that $y \in \bigcup_{k=1}^{\infty}(c_k, d_k)$ and choose an index p so that $y \in (c_p, d_p)$. Using both the inequalities $x < a_p < c_p < y$ and $b_p - a_p < 4(c_p - a_p)$ (see the definition of the intervals $[c_n, d_n]$), we have

$$\begin{aligned}|F(y) - F(x)| &= \left|\int_{c_p}^{y} f_p\right| \le \int_{c_p}^{y} |f_p| \le d_p - c_p \\ &= (b_p - a_p)^2 < 16(c_p - a_p)^2 \le 16(y - x)^2.\end{aligned}$$

We have thus shown that $|F(y) - F(x)| \le 16(y - x)^2$ for each $y \in (x, 1)$ and hence

$$\lim_{y \to x^+} \frac{F(y) - F(x)}{y - x} = 0.$$

A similar argument reveals that

$$\lim_{y \to x^-} \frac{F(y) - F(x)}{y - x} = 0$$

for each $x \in E \setminus \{0\}$. We conclude that $F'(x) = f(x)$ for all $x \in E$ and thus $F' = f$ on $[0, 1]$. Therefore, the function F' is a bounded derivative that is not Riemann integrable.

Goffman defines a continuous function $f_n \colon [0, 1] \to [0, 1]$ so that f_n is 0 on the intervals $[0, c_n]$ and $[d_n, 1]$, and $f_n((c_n + d_n)/2) = 1$. A proof that the corresponding function F satisfies $F' = f$ has a few extra details, but the reasoning is essentially the same. Discovering the necessary changes to the proof is a good (but nontrivial) exercise for students. For the record, Goffman's function F is increasing on $[0, 1]$. In our situation, the function $F(x) + x$ is increasing on $[0, 1]$; this function still has a bounded derivative that is not Riemann integrable on $[0, 1]$.

Construction of the open set O

We now make a few remarks concerning the set O. A set A of real numbers is open if for each $x \in A$ there exists $r > 0$ such that $(x - r, x + r) \subseteq A$. The simplest example of an open set is an open interval. It is easy to see that any union of open sets is an open set and that the intersection of two open sets is an open set. With a little more work, it can be shown that every open set is a countable union of disjoint open intervals (see [**1**, p. 66], [**2**, p. 350], or [**9**, p. 296]. A set D is dense in an interval $[a, b]$ if each open interval that intersects $[a, b]$ contains a point of D. The set $\mathbb{Q} \cap (a, b)$ of rational numbers in (a, b) is dense in the interval $[a, b]$. Since the set $\mathbb{Q}$ is countably infinite, the set $\mathbb{Q} \cap (0, 1)$ can be written as a sequence $\{r_k\}$ of distinct points. For example, we could write

$$\{r_k\} = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \ldots,$$

and continue the pattern of increasing denominators. For each positive integer k, let

$$I_k = \left(r_k - \frac{1}{2^{k+2}}, r_k + \frac{1}{2^{k+2}}\right) \cap (0, 1).$$

The set $O = \bigcup_{k=1}^{\infty} I_k$ is an open set, and since $\mathbb{Q} \cap (0, 1) \subseteq O$, the set O is dense in $(0, 1)$. We can then write $O = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where the intervals are pairwise disjoint and the number of open intervals is countably infinite (see below). For each positive integer n, let S_n be the set $\{k \in \mathbb{Z}^+ : I_k \subseteq (a_n, b_n)\}$. Since the sets S_n partition $\mathbb{Z}^+$ and $(a_n, b_n) = \bigcup_{k \in S_n} I_k$ for each n, we find that

$$\sum_{n=1}^{\infty} (b_n - a_n) \leq \sum_{n=1}^{\infty} \sum_{k \in S_n} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k) \leq \sum_{k=1}^{\infty} \frac{2}{2^{k+2}} = \frac{1}{2},$$

where $\ell(I_k)$ is the length of the interval I_k. This inequality, coupled with the fact that O is dense, shows that the number of intervals (a_n, b_n) is infinite. Hence, the set O has the properties needed for the construction of our function F.

For some people, it makes more sense to obtain the set O in the following manner. Choose an open interval $I_1 \subseteq (0, 1)$ that contains $1/2$ and has length less than $1/4$. Move down the list of rational numbers given above until you reach a number that is not in the interval I_1; call this term r_{k_1}. Choose an open interval $I_2 \subseteq (0, 1) \setminus I_1$ that contains r_{k_1} and has length less than $1/8$. Once again, move down the list of rational numbers until you reach a number that is not in the set $I_1 \cup I_2$; call this term r_{k_2}. Choose an open interval $I_3 \subseteq (0, 1) \setminus (I_1 \cup I_2)$ that contains r_{k_2} and has length less than $1/16$. Continuing in this fashion generates a sequence I_k of disjoint open intervals and the set $O = \bigcup_{k=1}^{\infty} I_k$ has the desired properties.

The set O can also be obtained as a complement of a set formed in much the same fashion as the well-known Cantor set. The set used in Volterra's example is often referred to as the Smith–Volterra–Cantor set (see [**3**]) and has Lebesgue measure $1/2$. Although these Cantor-like sets are quite interesting, they do introduce a new level of abstraction. The previous discussion illustrates one way to obtain a set O without bringing in these new ideas. In other words, a description of these Cantor-like sets is not essential to the construction of our example.

Comments for readers with a more advanced analysis background

We make one final comment for readers with more background in analysis. Using the function f that formed the basis for our example, let

$$V = \{x \in [0, 1] : f(x) \neq 0\} \subseteq \bigcup_{n=1}^{\infty} (c_n, d_n);$$

$$W = \{x \in [0, 1] : f(x) = 0\} = [0, 1] \setminus V.$$

It can be shown that each point of E is a point of density of the set W. To verify this, it is easiest to show that x is a point of dispersion of the set V. The details behind such an argument are essentially the same as those presented here (with the few extra details needed for Goffman's functions) to show that $F'(x) = 0$ for points in the set E. We just need to change the function f_n from the continuous function used earlier to the characteristic function of the interval (c_n, d_n). We leave the details to the interested reader. Consequently, the function f is approximately continuous at each point of E. Since f is continuous at each point of O, it follows that f is approximately continuous on [0, 1]. If F is then defined as $\int_0^x f$, where the integral is a Lebesgue integral, we find that $F' = f$ on [0, 1] by a standard theorem in the theory of Lebesgue integration (see [**8**, p. 227]).

Stromberg [**12**] states "Lebesgue said in his thesis (1902) that this example of Volterra's motivated him to devise a method of integration by which functions with bounded derivatives can be reconstructed from their derivatives." Lebesgue did in fact succeed in devising such an integration process. However, there are still derivatives that are not Lebesgue integrable; the function involving $x^2 \sin(1/x^2)$ on the interval [0, 1] provides one example. A search for an integration process that integrates all derivatives leads to some interesting ideas; see [**8**] for a discussion of the integrals that arise from this investigation.

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Summary. We present an example, different from Volterra's, of a bounded derivative that is not Riemann integrable. The existence of such functions was one of the motivations for Lebesgue to devise a stronger integration process. The goal of the presentation is to keep the ideas at a level appropriate for an undergraduate real analysis student.

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Generating Continuous Nowhere Differentiable Functions

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In this Note we show how one can associate a continuous and highly nondifferentiable function g to any bounded function f on a compact interval; the effect of this construction is that *more continuity for f implies less differentiability for g*. The technique we use is an adaptation of Billingsley and de Rham's proofs of nowhere differentiability of the Takagi–Van der Waerden's function (cf. [**1**], [**4**]).

The classical approach to the latter function is to use the properties of the distance function (and its iterates) to the nearest integer, cf. [**2**], [**5**]; instead, using Ledrappier's approach (cf. [**3**]), we will define g as an infinite series with respect to the Schauder base. More precisely, consider $f : [0, 1] \to \mathbb{R}$ and define $g : [0, 1] \to \mathbb{R}$ by

$$g = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} f(k \cdot 2^{-n}) \cdot e_{n,k}, \tag{1}$$

where $e_{n,k}$ are the standard Schauder functions defined by

$$e_{n,k}(x) = \begin{cases} 2^{n/2}x - \dfrac{k}{2^{n/2}} \text{ for } \dfrac{k}{2^n} \leq x \leq \dfrac{k}{2^n} + \dfrac{1}{2^{n+1}} \\ -2^{n/2}x + \dfrac{k+1}{2^{n/2}} \text{ for } \dfrac{k}{2^n} + \dfrac{1}{2^{n+1}} < x \leq \dfrac{k+1}{2^n} \\ 0 \text{ elsewhere,} \end{cases}$$

for $k = 0, 1, \ldots, 2^n - 1$ and $n = 0, 1, \ldots$. Their graphs are "little tents" of height $2^{-(1+n/2)}$ and base 2^{-n}. The choice of working with these piecewise linear functions is that they are nonzero on disjoint intervals. Note that g is well defined and, applying the standard Weierstrass M-test (cf. [**2**], [**5**]) to the uniform convergent sequence of Schauder functions in (1), it follows that g is continuous on $[0, 1]$.

We have the following result.

Theorem 1. *Consider a bounded function f as above. Then the function g defined by (1) is nondifferentiable at any continuity point x of f for which $f(x) \neq 0$.*

Taking into account that Riemann integrable functions are bounded and continuous almost everywhere (with respect to the Lebesgue measure), we deduce the following corollary.

Corollary 2. *If f is Riemann integrable, then g is nondifferentiable almost everywhere on the set $\{f \neq 0\}$.*

In particular, we obtain the following very nice result, the motivation of our Note.

Corollary 3. *If f is continuous, then g is nowhere differentiable on the set $\{f \neq 0\}$.*

Proof of Theorem 1. Let $x \in [0, 1)$ be a continuity point of f such that $f(x) \neq 0$. For $n \in \mathbb{N}$, we denote by x_n the largest dyadic $k \cdot 2^{-n}$, $k = 0, 1, \ldots, 2^n - 1$, smaller than or equal to x. Then there exist $\varepsilon > 0$ such that $|f(x)| \geq \varepsilon$ and, for sufficiently

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large n, $|f(x_n)| \geq \varepsilon$. Assume that g is differentiable at x. As $x_n \leq x < x_n + 2^{-n}$ and $x_n, x_n + 2^{-n} \to x$ as $n \to \infty$, we would have that the quotient ratio

$$\frac{g(x_n + 2^{-n}) - g(x_n)}{2^{-n}} \tag{2}$$

converges to a finite limit. For $n \geq 2$, let us denote

$$g_n = \sum_{m=0}^{n-1} \sum_{k=0}^{2^m - 1} f(k \cdot 2^{-m}) \cdot e_{m,k}.$$

As $e_{m,k}(x_n) = e_{m,k}(x_n + 2^{-n}) = 0$ for $m \geq n$ and $k = 0, 1, \ldots, 2^{m-1}$, we obtain

$$g(x_n + 2^{-n}) - g(x_n) = g_n(x_n + 2^{-n}) - g_n(x_n). \tag{3}$$

We also have

$$\begin{aligned} &\left|\left[g_n(x_n + 2^{-n}) - g_n(x_n)\right] - \left[g_{n-1}(x_n + 2^{-n}) - g_{n-1}(x_n)\right]\right| \\ &= \left|\sum_{k=0}^{2^{n-1}-1} f(k \cdot 2^{-(n-1)})\left[e_{n-1,k}(x_n + 2^{-n}) - e_{n-1,k}(x_n)\right]\right| \geq \varepsilon 2^{-(n+1)/2}, \end{aligned} \tag{4}$$

because, in the first line, all but one of the terms in the summation vanish and, in the second line, the maximum value corresponding to $e_{n-1,k}$ is $2^{-(n+1)/2}$. ■

A last ingredient we need is that

$$g_{n-1}(x_n + 2^{-n}) - g_{n-1}(x_n) = \frac{1}{2}\left[g_{n-1}\left(x_{n-1} + 2^{-(n-1)}\right) - g_{n-1}(x_{n-1})\right], \tag{5}$$

because the interval $[x_n, x_n + 2^{-n}]$ is contained in $[x_{n-1}, x_{n-1} + 2^{-(n-1)}]$ and each $e_{m,k}$ is linear on the latter interval for $m \leq n - 2$.

Combining (2)–(5), and applying (3) for both n and $n - 1$, we deduce that

$$\left|\frac{g(x_n + 2^{-n}) - g(x_n)}{2^{-n}} - \frac{g\left(x_{n-1} + 2^{-(n-1)}\right) - g(x_{n-1})}{2^{-(n-1)}}\right| \geq \varepsilon \cdot 2^{(n-1)/2},$$

which contradicts the convergence of the sequence in (2).

We excluded the case $x = 1$ (otherwise, the argument with the dyadic numbers would not hold); the latter can be treated similarly, by using the function $f(1 - x)$.

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Summary. In this note we show how can one associate a continuous and highly nondifferentiable function g to any bounded function f on a compact interval.

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A Function With Continuous Nonzero Derivative Whose Inverse Is Nowhere Continuous

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When students encounter continuous one-to-one functions in undergraduate real analysis courses, they are surprised to learn that the inverse function need not be continuous. Such functions have continuous inverses when defined on an interval [**1**], but not necessarily when the domain is not an interval. The standard way to illustrate this is to define a function on the union of two intervals.

Example 1. Let $f : (-\infty, 0) \cup [1, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x < 0; \\ x - 1, & \text{if } x \geq 1. \end{cases}$$

One can check that $f(x)$ is continuous, one-to-one, and onto. However,

$$f^{-1}(x) = \begin{cases} x, & \text{if } x < 0; \\ x + 1, & \text{if } x \geq 0 \end{cases}$$

is not continuous at $x = 0$.

If we remove all intervals from the domain, it's possible to define a function with a continuous nonzero derivative whose inverse is nowhere continuous, as given in Example 2 below. This does not contradict the inverse function theorem because our function is not defined on an open set.

Example 2. Let $X = A \cup B$, where $A = \{x \mid x \text{ is irrational and } x < 0\}$ and $B = \{x \mid x \text{ is rational and } x > 0\}$. Then, $f : X \to (0, \infty)$ defined by $f(x) = |x|$ is one-to-one and onto.

We will show that

$$f'(x) = \begin{cases} -1, & \text{if } x \in A; \\ 1, & \text{if } x \in B. \end{cases}$$

Let $x \in A$ and let $x_n \in A$ be a sequence converging to x and $x_n \neq x$. Then,

$$f'(x) = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = \lim_{n\to\infty} \frac{|\, x_n \,| - |\, x \,|}{x_n - x} = \lim_{n\to\infty} \frac{-x_n - (-x)}{x_n - x} = -1.$$

A similar argument shows that $f'(x) = 1$ for $x \in B$. It follows that $f'(x)$ is continuous. However, $f^{-1} : (0, \infty) \to X$ is given by

$$f^{-1}(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ -x & \text{if } x \text{ is irrational,} \end{cases}$$

which is not continuous at any point in $(0, \infty)$.

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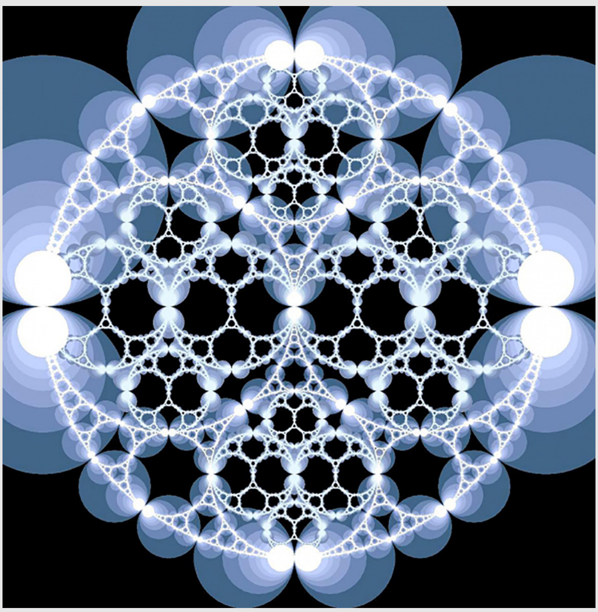
Summary. A function with continuous nonzero derivative whose inverse is nowhere continuous is given. The function satisfies all the premises of the inverse function theorem except one: it is not defined on an open set.

MARK LYNCH (MR Author ID: 233291) grew up in Brooklyn, earned a Ph.D. from LSU under the direction of Prof. Doug Curtis, and his current hobby is climbing up and rappelling down from the tops of tall pine trees.

Artist Spotlight
Anne Burns



Appolonian Gasket, Anne Burns; digital print, 2012. An Appolonian Gasket is constructed as follows: starting with any three mutually tangent circles the two circles that are tangent to the original three are added. At the next stage six new circles are added, each one tangent to three of the circles from the previous stage. Continuing in this pattern; at each stage, for every triple of circles, new circles tangent to each of the three are constructed.

See interview on page 375.

Anne Burns: Mathematical Botanist*

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Anne Burns is a retired mathematics professor who enjoys creating digital art influenced by mathematics and the natural world. We visited Anne near her home on Long Island, NY, in July 2016 to discuss her background and art. A portion of this interview appears below. Accompanying artwork appears on the following pages: 335, 339, 351, 363, and 374. Her artwork also appears on the covers of volumes 67(4), 70(3), and 75(2) and of this MAGAZINE.

Q: *Can you tell us about your educational background?*

AB: I hated math. I went through geometry in the tenth grade and my teacher was a southern lady who was about 80 years old, which I am now, but for a kid that wasn't too good. So I dropped out after that. When I went to college I majored in art. I went for one year, but my parents were not wealthy, so I couldn't afford to stay. I dropped out, got married, and later I went and worked for the government in Washington D.C. They gave me an aptitude test and sent me to take college algebra. I said,"This is silly. I don't even need to buy the book." I got a 40 on my first test and I said, "Well, I better buy the book!" I ended up loving it. I went on and took a calculus course and said, "Wow, this is it. I love calculus." So I just kept going. That was at George Washington University. We moved up to Massachusetts, and I took a couple of courses at Lowell Tech [now part of UMass Lowell] and then finally we moved to Long Island and I ended up going to C.W. Post and that's where I finished my bachelor's degree. Then I went to Stony Brook. My teachers told me, "Don't go to Stony Brook it's too hard. You should go to Adelphi." I said, "No, I want to go to Stony Brook." So I eventually ended up getting my Ph.D. there. I was 40 years old at the time.

Q: *Tell me about your intersection between art and math.*

AB: Well, I think the first thing that happened to me was around the middle of the 1980s. I started teaching at [C.W.] Post in 1976. Of course, there weren't many computers around in those days. I saw they were having a bunch of talks—I guess this was around 1987. It was a conference on fractals at NYU's Courant Institute. I went to it, and I was just overwhelmed. I said "this is unbelievable," because I have always been interested in art as well, and I didn't know there was any connection between mathematics and art. It was unbelievable. Mandelbrot spoke there and others; they had graphics that just blew my mind. Afterward, I bought a book on fractals and then I bought a computer. It was an early IBM computer with the little floppy disks and the screen was, I think, 320 pixels across and 200 up and down. There were three colors—white, green, and red on a black screen. I started making pictures right away. I got really interested in recursion and the idea of recursion.

At the same time that this was happening, my companion and I got interested in flowers. He was a professor of mine, and we'd walk around the neighborhood and I'd say "what flower is that?" and he was tremendously curious. As soon as I said that,

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MSC: Primary 01A70; Amy L. Reimann (MR Author ID: 1118776) and David A. Reimann (MR Author ID: 912704)

Figure 1 Mathscapes: Fractal flowers and landscapes

we'd be down at the library getting out a book on the identification of flowers. I would look at the structure of the flowers, the stems and the way they grew, and I would try to imitate that on a computer. It was pretty interesting, even with just three colors and very low resolution; using the idea of recursion was fantastic. Then I got really interested and took some courses in assembly language at one of the local colleges.

[See Figure 1.]

Q: *I know some of your art involved complex functions.*

AB: Yes, iterating systems of Möbius transformations, and also the subject of flows or complex functions as vector fields. I got interested in the physics of it, in particular flows. I made a lot of art out of that too. It evolved from my background in drawing because I found it so much fun to write mathematical programs that would draw pictures. It astonished me that you could do this. Because it's just amazing! You could take a bunch of zeros and ones and make a picture with it.

Q: *Can you talk about the process of how you work ?*

AB: I'll say, "Well maybe I'll try doing a particular plant structure." A lot of it is accident too; something evolves and you say, "Oh, I like that effect." So I'll keep it, and it's an evolutionary process to the actual programming stuff. You change a few variables and get something totally different. I don't ever have anything in mind when I start. Just experiment and see what comes up, keep working on it.

Q: *From a software standpoint what tools do you use?*

AB: Well, right now I'm using Processing. I have some covers on MATHEMATICS MAGAZINE. You'll see that they're very crude; the resolution is very low. These were back in the 1980s and 1990s. When Frank Farris was editor, I wrote my best article for MATHEMATICS MAGAZINE. It was on the Mandelbrot Set. (Plotting the Escape:

An Animation of Parabolic Bifurcations in the Mandelbrot Set, 75(2) (2002), pp. 104–116.)

Q: *How do you define the success of one of your artworks? How do you know when it's finished?*

AB: That's a good question actually. People have asked me that. I guess just when I like it I finally say, "OK, this is it. I'm not doing anymore." Or possibly when there is a deadline or if it's past a deadline. If it has to be in on Monday night at midnight, that's when it's finished. I did like the idea of telling kids that this is mathematics made visible. You can actually see certain properties that occur in mathematics. Now, of course, I just do it for recreation, just to see the beauty of it.

Q: *How long have you been submitting work to the Bridges organization?*

AB: My first Bridges was 2005 in Banff, which was the perfect place because of all the plants and flowers and mountains. It was interesting because I had done one of my landscapes—mathscapes I guess they were called—and I had exhibited it at that particular Bridges. I did some mountains in the background and water in the foreground. We got there and we went on a bus trip, and I was talking to Reza [Sarhangi,]. He said, "You got to go back and change the water color from blue to green in your picture." It turned out all the water in Banff is green.

Q: *Do you have a favorite piece that you've done?*

AB: Some of my earlier pieces I like better because I was trying to imitate nature. I was doing fractal scenery, mountains and clouds, and I developed algorithms for making cloudy skies and mountain ranges and flowers. I enjoyed doing that more than anything else. I still sort of favor them, although the resolution on them, because they're old, is not as good. I feel like calculus is more visual. I've always been attracted to the visual side of mathematics. I love the idea, for example, of the Möbius transformation, of taking a system with more than one transformation and taking an initial object and then transforming it into two objects. You could transform the object into more objects obviously but just to start with one into two and then each one of those two into two more is still really interesting.

Q: *What advice do you have for young mathematicians and artists?*

AB: That's a hard one. I just say learn as much as you can about every subject that you possibly can until you find something that you really love doing and then just do it.

Q: *Where are you from originally?*

AB: I was born in Brooklyn, New York. I lived there for six years. And then my family moved to Long Island. My father was a big baseball fan and he would take me into Ebbets Field to watch the ball game. I got in the habit of keeping the scores for him. Whenever he wasn't home, I would keep the score for the Dodgers. I'd write it all down—hits, errors, batting averages. In fact, my son Andrew, he hated long division, and then one summer, he started keeping track of the averages of the baseball players and that did it for him because he had to divide into three decimal places every time.

Q: *What did you want to be when you grew up?*

AB: Well, I wanted to be the first female player on the Brooklyn Dodgers. There's still no female players on the Brooklyn Dodgers. In fact, there are no Brooklyn Dodgers!

Q: *Are you an LA Dodgers fan?*

AB: No, I got very angry with them when they moved out to LA. I'm a Mets fan!

PROBLEMS

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Proposals

To be considered for publication, solutions should be received by May 1, 2017.

2006. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.*

Let $\mathcal{F}$ be a regular polygon whose n vertices $A_1, A_2, \ldots, A_n$ lie on the unit circle. For any point P on the plane of $\mathcal{F}$, let $f(P) = PA_1 \cdot PA_2 \cdot \ldots \cdot PA_n$ be the product of the distances from P to the vertices of $\mathcal{F}$. Find the maximum value of $f(P)$ over all points P lying in the interior of, or on any of the sides of $\mathcal{F}$. For which position(s) of P is this maximum attained?

2007. *Proposed by Ángel Plaza, Department of Mathematics, University Las Palmas de Gran Canaria, Spain.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Evaluate

$$\lim_{n\to\infty}\left(n \cdot \int_0^1 \left(\frac{2\left(x-\frac{1}{2}\right)^2}{x^2 - x + \frac{1}{2}}\right)^n f(x)\,\mathrm{d}x\right).$$

Math. Mag. **89** (2016) 378–385. doi:10.4169/math.mag.89.5.378.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Effective immediately, authors of proposals and solutions should send their contributions using the MAGAZINE'S *submissions system hosted at* http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We hope that this online system will help streamline our editorial team's workflow while still proving accessible and convenient to longtime readers and contributors. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2008. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Is there a function $f : \mathbb{R} \to (0, \infty)$ such that the inequality

$$\frac{f(x)}{f(y)} \leq |x - y|$$

holds for all real numbers x, y such that x is irrational and y rational?

2009. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

Consider any metric space (X, d). Let $\mathcal{F}_X$ be the collection of all functions $f : X \to \mathbb{R}$ (not necessarily continuous). The set $\mathcal{F}_X$ possesses natural operations of addition and multiplication, namely the sum $f + g$ and product fg of two elements f, g of $\mathcal{F}_X$ are characterized by the identities

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)g(x), \qquad \text{for all } x \in X.$$

Endowed with these operations, $\mathcal{F}_X$ is a ring. Since the sum and product of continuous real functions are continuous, the set $\mathcal{C}_X$ consisting of all continuous functions in $\mathcal{F}_X$ is a subring of $\mathcal{F}_X$. Is there a metric space (X, d) such that $\mathcal{C}_X$ isomorphic to $\mathcal{F}_X$ as rings, but $\mathcal{C}_X$ is a proper subset of $\mathcal{F}_X$?

2010. *Proposed by Mehtaab Sawhney (student), University of Pennsylvania, Philadelphia, PA.*

Let k be a positive integer. Consider the experiment of choosing a permutation π of k objects uniformly at random (i. e., any two permutations σ, π are equally likely to be chosen). Let N be the number of cycles of π. Find the expected value $\mathbf{E}[N2^N]$ of the random variable $N2^N$, as a function of k.

Quickies

Answers to the Quickies are on page 385.

1065. *Proposed by Adrian Chu (student), The Chinese University of Hong Kong, Hong Kong.*

Does the equation

$$a^2 + b^7 + c^{13} + d^{14} = e^{15}$$

have a solution in positive integers a, b, c, d, e?

1066. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Is there a real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose natural exponential

$$\exp(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \ldots$$

satisfies the equation $\exp(A) = \begin{pmatrix} e^a & e^b \\ e^c & e^d \end{pmatrix}$?

Solutions

A recursive sequence related to Egyptian fractions **October 2015**

1976. *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3$ and, for $n \geq 1$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$. Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{1+a_k}.$$

Editor's Note. Due to an editorial oversight, Problem 1976 is a duplicate of Problem 1952 (*Math. Mag.* **87** (2014) 292–298). A different solution appeared in *Math. Mag.* **88** (2015) 377–384.

Solution by Joel Schlosberg, Bayside, NY.
We prove by induction that

$$\sum_{k=1}^{n} \frac{1}{1+a_k} = \frac{1}{2} - \frac{1}{a_{n+1}-1} \tag{1}$$

holds for all $n \geq 1$. For $n = 1$, equation (1) holds because

$$\frac{1}{1+a_1} = \frac{1}{4} = \frac{1}{2} - \frac{1}{a_2 - 1}, \qquad \text{since } a_1 = 3 \text{ and } a_2 = (a_1^2+1)/2 = (3^2+1)/2 = 5.$$

If equation (1) holds for some $n \geq 1$, then

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{1+a_k} &= \frac{1}{2} - \frac{1}{a_{n+1}-1} + \frac{1}{1+a_{n+1}} = \frac{1}{2} - \frac{2}{a_{n+1}^2 - 1} = \frac{1}{2} - \frac{1}{\frac{1}{2}(a_{n+1}^2+1) - 1} \\ &= \frac{1}{2} - \frac{1}{a_{n+2}-1},\end{aligned}$$

showing that equation (1) holds for $n+1$. This completes the inductive step of the proof.

Next, we prove by induction that

$$a_n \geq 2n+1 \qquad \text{for all } n \geq 1. \tag{2}$$

The inequality holds for $n = 1$ since $a_1 = 3 = 2 \cdot 1 + 1$. If the inequality in (2) holds for some $n \geq 1$, then certainly $a_n - 1 \geq 2n \geq 2 \cdot 1 = 2$, so

$$a_{n+1} = \frac{a_n^2+1}{2} = a_n + \frac{(a_n-1)^2}{2} \geq a_n + \frac{2^2}{2} = a_n + 2,$$

hence $a_{n+1} \geq a_n + 2 \geq (2n+1) + 2 = 2(n+1)+1$ by the inductive assumption. This completes the proof of (2).

From (2), we have $\lim_{n\to\infty} a_n = \infty$. Thus, using (1),

$$\sum_{k=1}^{\infty} \frac{1}{1+a_k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+a_k} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{a_{n+1}-1}\right) = \frac{1}{2}.$$

Also solved by Adnan Ali (India), Michael Arakelian (Armenia), Michel Bataille (France), Armstrong Problem Solvers (Armstrong State University), Joseph DiMuro, Peter Hauber (Germany), Eugene A. Herman, Hofstra University Problem Solvers, FAU Problem Solving Group, James Magliano, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Paolo Perfetti (Italy), Amol Sasane (UK), Achilleas Sinefakopoulos (Greece), Nicholas C. Singer, Skidmore College Problem Group, Allen Stenger, Michael Vowe (Switzerland), and the proposer. There were nine incomplete or incorrect solutions.

A property of isosceles inscribable tetrahedra **October 2015**

1977. *Proposed by Marcel Chirita, Bucharest, Romania.*

Let $ABCD$ be a tetrahedron inscribed in a sphere $\mathcal{S}$ of radius R. For every point M in space, define $f(M) = AM^2 + BM^2 + CM^2 + DM^2$. Suppose that $f(M)$ is constant for all points M on $\mathcal{S}$.

(a) Calculate $f(M)$.

(b) Prove that $AB = CD$, $AC = BD$, and $AD = BC$; that is prove that $ABCD$ is an isosceles tetrahedron.

Solution by Armstrong Problem Solvers, Armstrong State University, GA.

(a) We prove that $f(M) = 8R^2$ for all points M on $\mathcal{S}$. Let O be the center of $\mathcal{S}$, and for each point M in space, denote by $\overrightarrow{M}$ the vector from O to M. If P and M are any two points on $\mathcal{S}$, then

$$PM^2 = \left|\overrightarrow{M}\right|^2 - 2\overrightarrow{M} \cdot \overrightarrow{P} + \left|\overrightarrow{P}\right|^2 = 2R^2 - 2\overrightarrow{M} \cdot \overrightarrow{P}.$$

Thus, for every point M on $\mathcal{S}$,

$$\begin{aligned} f(M) &= AM^2 + BM^2 + CM^2 + DM^2 = 8R^2 - 2\overrightarrow{M} \cdot \left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}\right) \\ &= 8R^2 - 2\overrightarrow{M} \cdot \mathbf{v}, \qquad \text{where } \mathbf{v} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}. \end{aligned} \tag{1}$$

Using the assumption that $f(M)$ is constant for all M on $\mathcal{S}$, we prove that $\mathbf{v} = \mathbf{0}$. If $\mathbf{v} \neq \mathbf{0}$, let $\mathbf{w} = R\mathbf{v}/|\mathbf{v}|$. Clearly, $|\mathbf{w}| = R$, so $\mathbf{w} = \overrightarrow{W}$ is the vector from O to some point W on $\mathcal{S}$. The antipodal point W^* to W on $\mathcal{S}$ satisfies $\overrightarrow{W^*} = -\mathbf{w}$. Since f is constant on $\mathcal{S}$, we have $f(W) = f(W^*)$, hence from equation (1):

$$8R^2 - 2\mathbf{w} \cdot \mathbf{v} = 8R^2 - 2(-\mathbf{w}) \cdot \mathbf{v} \quad \Rightarrow \quad 0 = \mathbf{w} \cdot \mathbf{v} = \frac{R\mathbf{v}}{|\mathbf{v}|} \cdot \mathbf{v} = \frac{R\,|\mathbf{v}|^2}{|\mathbf{v}|} = R\,|\mathbf{v}| \quad \Rightarrow \quad \mathbf{v} = \mathbf{0}.$$

Thus, $\mathbf{v} \neq \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$: contradiction. This proves that $\mathbf{v} = \mathbf{0}$. Equation (1) now shows that the constant value of f on $\mathcal{S}$ is $8R^2$ as claimed above.

(b) We prove that $ABCD$ is isosceles. Since A, B, C, D lie on $\mathcal{S}$, we have $f(A) = f(B) = f(C) = f(D)$, hence $0 = f(A) + f(B) - f(C) - f(D) = 2(AB^2 - CD^2)$, as follows from direct calculation. Thus, $AB = CD$. Applying the same argument to the tetrahedra $ACBD$ and $ADBC$, we obtain $AC = BD$ and $AD = BC$, also proving that $ABCD$ is isosceles.

Also solved by Michel Bataille (France), Dmitry Fleischman, Mowaffad Hajja (Jordan), Hidefumi Katsuura & Edward Schmeichel, Angel Plaza & José M. Pacheco (Spain), Amol Sasane (United Kingdom), Joel Schlosberg, Michael Vowe (Switzerland), and the proposer. There was one incomplete or incorrect solution.

A limit of n-norm integrals **October 2015**

1978. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Evaluate

$$\lim_{n\to\infty}\left[\int_1^{e^2}\left(\frac{\ln x}{x}\right)^n dx\right]^{1/n}.$$

Editor's Note. Due to an editorial oversight, Problem 1978 is a duplicate of Problem 1954 (*Math. Mag.* **87** (2014) 292–298). A different solution appeared in *Math. Mag.* **88** (2015) 377–384.

Solution by Robert A. Agnew, Buffalo Grove IL & Palm Coast FL.
The limit has the value e^{-1}. Let $f(x) = (\ln x)/x$ for $x \geq 1$, and

$$G_n = \left[\int_1^{e^2}\left(\frac{\ln x}{x}\right)^n dx\right]^{1/n} = \left[\int_1^{e^2}(f(x))^n dx\right]^{1/n} \qquad \text{for } n \geq 1.$$

Since $f'(x) = x^{-2}\ln(e/x)$, we have $f'(x) > 0$ for $x < e$, while $f'(x) < 0$ for $x > e$, and $f'(e) = 0$; thus, f attains its maximum value $f(e) = (\ln e)/e = e^{-1}$ at $x = e$. Hence, for $n \geq 1$,

$$G_n \leq \left[\int_1^{e^2} e^{-n} dx\right]^{1/n} = e^{-1}(e^2-1)^{1/n} =: R_n.$$

On the other hand, $f(x) \geq 0$ for $x \geq 1$, and $f(x) \geq 1/x$ for $x \geq e$, so

$$G_n \geq \left[\int_e^{e^2} x^{-n} dx\right]^{1/n} = e^{-(1-\frac{1}{n})}\left[\frac{1-e^{-(n-1)}}{n-1}\right]^{1/n} =: L_n.$$

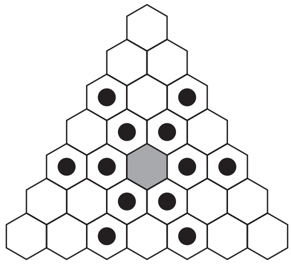
Thus, $L_n \leq G_n \leq R_n$ for all $n \geq 1$. It is straightforward to verify that $L_n \to e^{-1}$ and $R_n \to e^{-1}$ as $n \to \infty$. By the squeeze theorem, we also have $G_n \to e^{-1}$ as $n \to \infty$.

Also solved by Michel Bataille (France), Gerald E. Bilodeau, Chris Boucher, Shreya Dalal & Tommy Goebeler, Ross Dempsey, Dmitry Fleischman, GWstat Problem Solving Group, G.R.A.20 Problem Solving Group (Italy), Eugene A. Herman, Charles Ross McCarthy, Moubinool Omarjee (France), Paolo Perfetti (Italy), Pittsburg State University Problem Solving Group, Ángel Plaza (Spain), Henry Ricardo, Joel Schlosberg, Achilleas Sinefakopoulos (Greece), Nicholas C. Singer, Michael Vowe (Switzerland), Haohao Wang & Jerzy Wojdylo, and the proposer. There were two incomplete or incorrect solutions.

Hex chess rooks come in all colors **October 2015**

1979. *Proposed by Dan Ullman, George Washington University, Washington, DC and Stan Wagon, Macalester College, Saint Paul, MN.*

What is the chromatic number of G_n, the graph whose vertices are the cells in a triangular grid of hexagons and whose edges correspond to two cells in the same row of adjacent cells in any of the three directions? The diagram shows G_7, with the neighbors of one vertex marked.



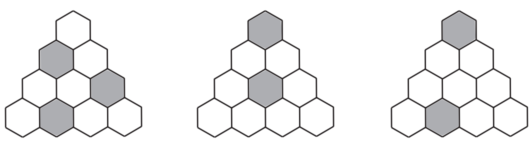
This graph can be viewed as the graph of moves of a rook in hexagonal chess (http://en.wikipedia.org/wiki/Hexagonal_chess).

Solution by Rob Pratt, Washington, DC.
Let χ_n be the chromatic number of G_n. We show that

$$\chi_n = \begin{cases} n+1 & \text{if } n = 2 \text{ or } n = 4, \\ n & \text{otherwise.} \end{cases} \tag{1}$$

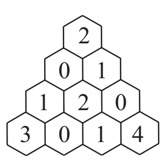
Evidently, we have $\chi_1 = 1$ and $\chi_2 = 3$, in agreement with expression (1).

Since the bottom row of G_n has n pairwise adjacent vertices, we have $\chi_n \geq n$. For $n \geq 2$, G_n has six symmetries corresponding to the isometries of the triangular grid of hexagons. By inspection, G_4 has 11 maximal independent vertex sets: two of size 3 (of type 1 in the figure below, modulo symmetry) and nine of size 2 (three of type 2 and six of type 3, given by the figure below modulo symmetry).



type 1 type 2 type 3

Since a monochromatic vertex set must necessarily be independent, a hypothetical four-coloring of the 10 vertices of G_4 would necessarily have monochromatic vertex sets of sizes 3, 3, 2, 2, of which the three-vertex sets are exactly those of type 1. There remain two colors for the remaining four vertices, namely the central hexagon and three extremal ones. However, the three extremal hexagons correspond to pairwise-adjacent vertices of G_4, so coloring them requires three more colors. We conclude that a four-coloring of G_4 does not exist. The figure below shows a five-coloring of G_4 (with colors $\{0, 1, 2, 3, 4\}$).



For odd n, we have the following n-coloring with colors $\{0, 1, \ldots, n-1\}$ (mod n):

$$\begin{array}{ccccccccccc}
 & & & & & n-2 & & & & & \\
 & & & & n-4 & & n-3 & & & & \\
 & & & n-6 & & n-5 & & n-4 & & & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & & \\
 & 2 & & 3 & & \cdots & & n-1 & & 0 & \\
0 & & 1 & & \cdots & & \cdots & & n-2 & & n-1
\end{array} \tag{2}$$

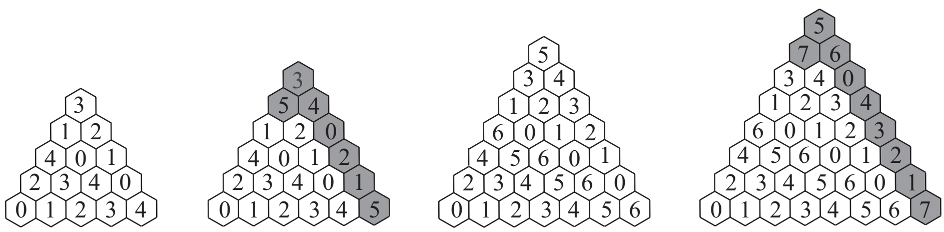
The entries increase by 1 (mod n) in two directions ($\rightarrow$ and $\nwarrow$), increase by 2 (mod n) in the third direction ($\nearrow$) and do not repeat in any direction because n is odd.

For even $n \geq 6$, we construct an n-coloring starting with the $(n-1)$-coloring of G_{n-1} above (obtained upon replacing n in (2) by $n-1$), changing the color of the top vertex to $n-1$, and appending a row to the right, as follows (the new entries are underlined):

$$\begin{array}{ccccccccccccc}
 & & & & & & \underline{n-3} & & & & & & \\
 & & & & & \underline{n-1} & & \underline{n-2} & & & & & \\
 & & & & n-5 & & n-4 & & \underline{0} & & & & \\
 & & & n-7 & & n-6 & & n-5 & & \underline{n-4} & & & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & & \underline{\cdots} & & \\
 & 2 & & 3 & & \cdots & & n-2 & & 0 & & \underline{1} & \\
0 & & 1 & & \cdots & & \cdots & & n-3 & & n-2 & & \underline{n-1}
\end{array}$$

The underlined omitted entries $\underline{\ldots}$ above are $2, 3, \ldots, (n-5)$ (mod n), increasing in the northwest direction ($\nwarrow$).

The figure below shows the colorings for $n = 5, 6, 7, 8$.



Also solved by Kenneth E. Cherasia & Tiffany M. Callanan, Joseph DiMuro, Florida Atlantic University Problem Solving Group and the proposer. There was 1 incomplete or incorrect solution.

Avoiding the integer lattice **October 2015**

1980. *Proposed by H. A. ShahAli, Tehran, Iran.*

For every S subset of the plane, let $\operatorname{diam}(S) = \sup\{\|x - y\| : x, y \in S\}$. Let $n \geq 1$ be an integer and $S_1, S_2, \ldots, S_n$ subsets of the plane such that $\sum_{k=1}^{n} \operatorname{diam}(S_k) < \sqrt{2}$. Define $S = \bigcup_{k=1}^{n} S_k$. Prove that there is a translation of S that avoids all points with integer coordinates. That is, prove that there are real numbers r and s such that $((r, s) + S) \cap (\mathbb{Z} \times \mathbb{Z}) = \emptyset$.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA, and the editors.
Lemma: Given $\delta > 0$ and a finite collection $\{I_1, I_2, \ldots, I_n\}$ of closed intervals of $\mathbb{R}$ with $\sum_{k=1}^{n} \operatorname{diam}(I_k) < \delta$, there exists a translation of $J = \bigcup_{k=1}^{n} I_k$ that avoids all integer multiples of δ. (For a closed interval $I = [a, b]$, we denote by $\operatorname{diam}(I) = b - a$ its length.)

Proof. Without loss of generality, we may assume $\delta = 1$ by homogeneity. Thus, we show that some translation of J avoids all integers. Also without loss of generality, we may replace any point $x \in J$ with its fractional part $\rho(x) = x - \lfloor x \rfloor \in [0, 1)$ because $\rho(x)$ differs from x by an integral translation. Thus, it suffices to show that $K = \bigcup_{k=1}^{n} \rho(I_k)$ is a proper subset of $[0, 1)$, because any $r \in [0, 1) \setminus K$ will yield a translation $K - r$ of K, hence $J - r$ of J, that avoids all integers. Note that, for $k = 1, \ldots, n$, the set $\rho(I_k) \subset [0, 1)$ is either a closed interval $[a_k, b_k]$ or a union of two intervals $[0, b_k] \cup [a_k, 1)$. It is clear that the (sum of the) length(s) of the interval(s) in $\rho(I_k)$ is at most $\operatorname{diam}(I_k)$. Since $\sum_{k=1}^{n} \operatorname{diam}(I_k) < 1$, K is a union of subintervals of $[0, 1)$ the sum of whose lengths is strictly less than one. Hence, K is a proper subset of the length-1 interval $[0, 1)$, concluding the proof. ■

Continuing the solution, replace the sets $S_1, \ldots, S_n$ with their convex hulls (note that doing so does not change their diameters). It suffices to solve the problem in this case. The orthogonal projection S^+ of S on the line L^+ with equation $y = x$ is a union of n closed intervals $I_1, \ldots, I_n$ of L^+ with lengths at most $\operatorname{diam}(S_1), \ldots, \operatorname{diam}(S_n)$, hence $\sum_{k=1} \operatorname{diam}(I_k) \leq \sum_{k=1}^{n} \operatorname{diam}(S_k) < \sqrt{2}$. Using the lemma, we find $u \in \mathbb{R}$ so that the translation $S^+ + (u, u)$ of S^+ along L^+ avoids all lattice points (m, m) $(m \in \mathbb{Z})$ on L^+ (i. e., all integer multiples of the length-$\sqrt{2}$ vector $(1, 1)$). Repeating the preceding argument for the line L^- with equation $y = -x$, we obtain $v \in \mathbb{R}$ such that the translation $S^- + (v, -v)$ of the orthogonal projection S^- of S on L^-avoids all lattice points $(m, -m)$ on L^-.

It follows from the choice of u that $S' = S + (u + v, u - v) = S + (u, u) + (v, -v)$ avoids the lines with equations $x + y = 2m$ for $m \in \mathbb{Z}$, while the choice of v ensures that S' avoids the lines $x - y = 2m$. On the other hand, $\mathbb{Z} \times \mathbb{Z}$ is a subset of the union of the lines with equations $x - y = 2m$, $x + y = 2m + 1$ for $m \in \mathbb{Z}$, so the translation $S' + (1/2, 1/2) = S + (u + v + 1/2, u - v + 1/2)$ avoids $\mathbb{Z} \times \mathbb{Z}$.

Also solved by the proposer. There was one incomplete or incorrect solution.

Answers

Solutions to the Quickies from page 379.

A1065. The least common multiple of 2, 7, 13 and 14 is 182, which is coprime to 15. We have $7 \cdot 182 + 1 = 85 \cdot 15 = 1275$. Thus,

$$(4^{637})^2 + (4^{182})^7 + (4^{98})^{13} + (4^{91})^{14} = 4 \cdot 4^{1274} = 4^{1275} = (4^{85})^{15}.$$

A1066. There is no such matrix A. If λ_1, λ_2 are the (possibly nonreal) eigenvalues of A, then $e^{\lambda_1}, e^{\lambda_2}$ are the eigenvalues of $\exp(A)$. Thus, for any A satisfying the equation in the statement, we have

$$e^{a+d} = e^{\operatorname{tr}(A)} = e^{\lambda_1+\lambda_2} = e^{\lambda_1}e^{\lambda_2} = \det(\exp(A)) = e^a e^d - e^b e^c = e^{a+d} - e^{b+c}.$$

This implies that $e^{b+c} = 0$, which is impossible.

Editor's Note. Since $e^z = 0$ has no solutions for complex z, the argument above shows that no complex 2×2 matrix A satisfies the equation in the statement.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Royal Swedish Academy of Sciences, The Nobel Prize in Physics 2016 Popular Science Background: Strange phenomena in matter's flatlands, https://www.nobelprize.org/nobel_prizes/physics/laureates/2016/popular-physicsprize2016.pdf.

The 2016 Nobel Prize in Physics was awarded for "theoretical discoveries of topological phase transitions and topological phases of matter." The Academy offers a Scientific Background document that will scare off all but quantum physicists, and newspaper accounts have focused on the too-familiar "equivalence" of donuts and coffee cups; but this "popular" account is informative while being digestible. Apart from the well-known phases of matter (solid, liquid, gas, and plasma), matter at very low temperatures in strong magnetic fields is now known to take on "more exotic states," such as "quantum condensates," with unusual properties (e.g., superconductivity). At such low temperatures, atoms (or electrons) group into pairs of vortices. With a rise in temperature or a reduction in magnetic field, a phase transition occurs (analogous to ice-to-water): The vortices wander away from each other, and the conductance of the material changes—but only in quantized steps. Hence, the analogy to holes in familiar manifolds. The connection is that the patterns of vorticity are topologically distinct and characterized by topological invariants (Chern numbers). This year gives you the opportunity to inform friends, neighbors, and students who inquire about the award just a little bit about the nature of topology and the mathematically important concept of invariance—of course, a donut and a coffee cup will be inevitable props for your discussion.

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon, *The Joy of SET: The Family Game of Visual Perception*, Princeton University Press, 2016; xii+306 pp, $29.95. ISBN 978-0-691-16614-8.

Clark, David, George Fisk, and Nurullah Goren, A variation on the game SET, *Involve* 9 (3) (2016) http://faculty.gvsu.edu/clarkdav/ugr/antiset.pdf.

The card game of SET (released in 1991) involves recognizing different kinds of patterns ("*SET*s") in groups of three cards. But there is much more to it than simple set theory. There is now an entire book on the mathematical smorgasbord that the game engenders: affine planes, modular arithmetic, combinatorics (how many *SET*s are there in the 81 cards?), probability, algorithms (to determine if there are no *SET*s in a given configuration), linear algebra, error-correcting codes, simulations, and the normal distribution (courtesy of the central limit theorem). There are exercises (with solutions) and projects. The book can be read at various mathematical levels and should have wide appeal among devotees of the game. A separate article uses affine geometries to find a first-player win for two-player "Anti-SET," a misère variation in which the objective is to avoid drawing *SET*s: Players take turns selecting a card from the entire remaining SET deck; the first player to hold a *SET* loses.

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Henshaw, John M. *An Equation for Every Occasion: Fifty-Two Formulas and Why They Matter*, Johns Hopkins University Press, 2014; x+187 pp, \$29.95, \$27.95(P). ISBN 978-1-4214-1491-1, 978-1-4214-1983-1.

Here you will find Hooke's law, the Doppler effect, Newton's law of gravitation, the Stefan–Boltzmann law, and lots more. What is remarkable are connections that author Henshaw makes between an equation and everyday life: The von Bertalannfy equation for length of a creature begins a discussion of Asian carp, kudzu, and zebra mussels; the Hagen–Poiseuille equation for fluid flow brings on an experiment with pitch that started in 1927 and is still continuing, and an equation for rock fracture leads into how hydraulic fracturing (for oil and gas) works. These "equation story" vignettes run three or four pages and are all entertaining as well as informative.

Takahashi, Shin, Iroha Inoue, and Trend-Pro Co., Ltd., *The Manga Guide to Regression Analysis*, No Starch Press, 2016; xii+220 pp, \$24.95. ISBN 978-1-59327-728-4.

Still in print is the 1993 pre-manga *Cartoon Guide to Statistics* by Larry Gonick and Woollcott Smith. While that book has no prerequisites and covers most of introductory statistics, this manga book starts with a "general math refresher" chapter that reprises algebra, logs and exponentials, differentiation, and matrix operations before a briefer review of variance, probability density functions, and distributions (normal, chi-squared, and F). The context is that a manager of a tea shop is teaching a server how to do regression analysis so as to predict sales. A lot is packed in: confidence intervals, prediction intervals, standardized residuals, analysis of variance, multiple regression analysis, adjusted R^2, partial regression coefficients, Mahalanobis distance, logistic regression, maximum likelihood function, and adjusted odds ratio. An appendix gives instructions for use of a spreadsheet, downloadable from the book's website, that contains the data used in the book.

Field, Andy, *An Adventure in Statistics: The Reality Enigma*, Sage, 2016; xvii+746 pp, \$56(P). ISBN 978-1-44621045-1.

What happens when a professor of psychopathology writes a statistics textbook? What you get is the most unusual statistics textbook that I have ever encountered (beautifully packaged, too). It includes "an attempt to make probability theory interesting by using bridges of death and skulls instead of coin tossing" (p. 226). The statistics material is woven into a sci-fi fantasy story that is accompanied by gorgeous full-color illustrations and cartoons. Like other statistics texts, the book is heavy, but it is weighted down by the story line, not by hundreds of exercises nor multiple examples of the same calculations. A companion website features learning objectives, videos, solutions to the "puzzles" (exercises), the datasets, R scripts for a few chapters, multiple-choice questions (with answers), and flashcards. A question: If the course doesn't finish the whole book by the end of the semester, will students read on to find how the story ends? More importantly, will this unusual and entertaining book help motivate and hold students' interest in statistics?

Consortium for Mathematics and Its Applications (COMAP) and the Society for Industrial and Applied Mathematics (SIAM), *GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education*, 2016; 216 pp, \$20(P). ISBN 978-1-611974-43-0. Free download at http://www.siam.org/reports/gaimme.php.

This book outlines for teachers what mathematical modeling is, making a key distinction between "word problems"—e.g., an addition problem clothed with objects (apples) and labels (Jim and Suzy)—and mathematical modeling, which is more open-ended with context to take into account. The authors argue for doing mathematical modeling at all grade levels and offer rubrics for assessing modeling reports. (Disclosure: I edit a journal devoted to mathematical modeling that is published by COMAP.)

Correction: The standard Frame-Stewart algorithm for four-peg Tower of Hanoi, discussed in the February 2016 column, has indeed been shown to be optimal, by Thierry Bousch (La quatrième tour de Hanoï, *Bulletin of the Belgian Mathematical Society Simon Stevin* 21 (5) (2014) 895–912). (Thanks to Donald Knuth, who calls the solution "quite brilliant.")

NEWS AND LETTERS

57^{Th} International Mathematical Olympiad

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Problems (Day 1)

1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that lines BD, FX, and ME are concurrent.
2. Find all positive integers n for which each cell of an $n \times n$ table can be filled with one of the letters I, M, and O in such a way that:
   - In each row and column, one-third of the entries are I, one-third are M, and one-third are O; and
   - In any diagonal, if the number of entries on the diagonal is a multiple of three, then one-third of the entries are I, one-third are M, and one-third are O.

   Note: The rows and columns of an $n \times n$ table are each labeled 1 to n in a natural order. Thus, each cell corresponds to a pair of positive integers (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of the first type consists of all cells (i, j) for which $i + j$ is a constant, and a diagonal of the second type consists of all cells (i, j) for which $i - j$ is a constant.
3. Let $P = A_1A_2 \ldots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

Problems (Day 2)

4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible value of the positive integer b such that there exists a nonnegative integer a for which the set

$$\{P(a + 1), P(a + 2), \ldots, P(a + b)\}$$

   is fragrant?

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5. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2,016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4,032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

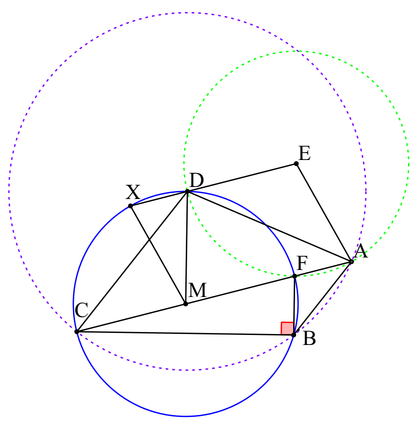
6. There are $n \geq 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

   (a) Prove that Geoff can always fulfill his wish if n is odd.

   (b) Prove that Geoff can never fulfill his wish if n is even.

Solutions There are six problems and six members of the USA International Mathematical Olympiad team. In honor of our students' historic performance, we have invited each of them to submit their solution and perspective, modified and polished with hindsight.

1. (Solution written by Ashwin Sah.)



First, $\angle EDA = \angle EAD = \angle FAD$, which immediately shows that $DE \parallel AC$. Thus, D, E, X are collinear. Notice

$$\angle ABC = 90 + \angle ABF = 90 + \angle FAB = 90 + \angle DAC,$$

and since triangle DAC is isosceles with vertex D, we immediately find $\angle ADC = 360 - 2\angle ABC$. Now D is opposite line AC from B, $DA = DC$, and $\angle ADC = 360 - 2\angle ABC$. There is a unique point that satisfies all these conditions, because it must be on the perpendicular bisector of AC, and the angle condition limits it to two positions, while the orientation constraint brings it down to only one. And furthermore, the circumcenter of triangle ABC also satisfies these conditions! Indeed, since $\angle ABC$ is obtuse, this circumcenter is opposite AC from B, which implies the same angle condition and length equality. Therefore, D is the circumcenter of triangle ABC.

Now,

$$\angle DBF = \angle DBA - \angle FBA = \angle DAB - \angle FBA,$$

and

$$\angle DAB - \angle FBA = \angle DAB - \angle FAB = \angle DAC = \angle DCF.$$

Therefore, $\angle DBF = \angle DCF$, and $BCDF$ is a cyclic quadrilateral. But since M is the midpoint of the hypotenuse in right triangle BCF, clearly, M is the center of this circle. And now

$$\angle MDX = \angle DMF = 2\angle DCA = 2\angle DAC$$

while

$$\angle MXD = \angle MXE = \angle MAE = 2\angle DAC,$$

so $\angle MDX = \angle MXD$, and thus, $MX = MD$. So X is on the circle with center M passing through B, C, D, F as well.

Finally, we have $\angle MXD = \angle MDX = 2\angle DAC$ while $\angle MBF = \angle MFB = 2\angle FAB = 2\angle DAC$ so that triangles MXD and MFB are congruent. Thus, arcs DX, BF are congruent in size, and thus, $BXDF$ is an isosceles trapezoid with $BX \parallel DF$. And of course, BD and FX intersect on its line of symmetry. Now, we repeat the trick above.

$$\angle MCD = \angle MDC = \angle DAC,$$

so

$$360 - 2\angle AFD = 2\angle DFC = \angle DMC = 180 - 2\angle DAC$$

while

$$\angle DEA = 180 - 2\angle EAD = 180 - 2\angle DAC.$$

So $\angle AED = 360 - 2\angle BFD$, $EA = ED$, and E is opposite line AD from F. Thus, similar to above, E is the circumcenter of triangle AFD. Then $ED = EF$.

Remembering $MD = MF$, we find that line ME is the perpendicular bisector of segment DF and, thus, is the line of symmetry of isosceles trapezoid $BXDF$. Thus, it is obvious why ME passes through the intersection of BX, DF! We are done. □

This problem was proposed by Belgium.

2. (Solution written by Michael Kural.)

Clearly, n must be divisible by 3. We claim the table can be filled if and only if $9 \mid n$. First, note that we can fill out a 9×9 table in a valid way as follows:

I	*I*	*I*	*M*	*M*	*M*	*O*	*O*	*O*
M	*M*	*M*	*O*	*O*	*O*	*I*	*I*	*I*
O	*O*	*O*	*I*	*I*	*I*	*M*	*M*	*M*
O	*O*	*O*	*I*	*I*	*I*	*M*	*M*	*M*
I	*I*	*I*	*M*	*M*	*M*	*O*	*O*	*O*
M	*M*	*M*	*O*	*O*	*O*	*I*	*I*	*I*
M	*M*	*M*	*O*	*O*	*O*	*I*	*I*	*I*
O	*O*	*O*	*I*	*I*	*I*	*M*	*M*	*M*
I	*I*	*I*	*M*	*M*	*M*	*O*	*O*	*O*

Now, given a positive integer k, we can overlay k^2 copies of this 9×9 table onto a $9k \times 9k$ table. This clearly preserves the validity of both conditions, so if n can be written in the form $9k$, an $n \times n$ table can be filled out as desired.

Conversely, suppose a table satisfying the given conditions has been filled out. The table can be characterized by ordered pairs of integer coordinates (x, y) with $1 \leq x, y \leq n$. For $1 \leq k \leq n$, let A_k, B_k denote the multisets of I's, M's, and O's in the rows and columns given by $x = k$ and $y = k$, respectively. Let C_k denote the diagonal $x + y = k$ for $2 \leq k \leq 2n$, and let D_k denote the diagonal $x - y = k$ for $-n + 1 \leq k \leq n - 1$.

We define the multiset unions

$$A' = \bigcup_{k \equiv 2 \pmod 3} A_k$$

$$B' = \bigcup_{k \equiv 2 \pmod 3} B_k$$

$$C' = \bigcup_{k \equiv 1 \pmod 3} C_k$$

$$D' = \bigcup_{k \equiv 0 \pmod 3} D_k.$$

Additionally, define the multiset R as the union of the entire table, S as the union of all cells (x, y) with $x, y \equiv 2 \pmod 3$, and

$$T = A' \cup B' \cup C' \cup D'.$$

The number of times a cell (x, y) is repeated in T depends only on the residues left by x, y modulo 3. A' contributes a copy of each of the pairs of residue classes (2, 0), (2, 1), (2, 2). Similarly, B' contributes copies of (0, 2), (1, 2), (2, 2), C' contributes copies of (0, 1), (1, 0), (2, 2), and D' contributes copies of (0, 0), (1, 1), (2, 2). In total, each pair of residue classes is represented exactly once, except for (2, 2), which is represented four times. Thus,

$$T = R \cup S \cup S \cup S$$

We call a multiset balanced if it contains equal numbers of I, M, and O. Note that A', B', C', and D' are balanced, so T is balanced as well. But considering the union of all rows implies that R is also balanced, so finally S must be balanced. But a balanced multiset must have cardinality divisible by 3, and as the cardinality of S is $\left(\frac{n}{3}\right)^2$, we have

$$3 \,\Big|\, \left(\frac{n}{3}\right)^2,$$

implying $9 \mid n$, as desired. ■

This problem was proposed by Australia.

3. (Solution written by Ankan Bhattacharya.)

Note that it suffices to prove the statement when $n = p^e$, an odd prime power. We proceed by strong induction on k.

Suppose $k = 3$, and without loss of generality, let the vertices of P be (0, 0), (a, b), and (c, d). Then, we have

$$a^2 + b^2, \quad c^2 + d^2, \quad \text{and} \quad (a - c)^2 + (b - d)^2$$

are all divisible by p^e. Now, the right-hand side of the identity

$$(ad - bc)^2 = (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2$$

is divisible by p^{2e}, so the left-hand side is as well. This means that $|ad - bc|$ is divisible by p^e. However, this is equal to $2S$ by the shoelace formula, so $2S$ is divisible by n.

For every $a \in \mathbb{N}$, define $v_p(\pm\sqrt{a}) = \frac{1}{2}v_p(a)$. We define $v_p(0) = \infty$.

Lemma. If a, b, and c satisfy $a^2, b^2, c^2 \in \mathbb{N}_0$ and $a + b + c = 0$, then the smallest two of $v_p(a)$, $v_p(b)$, and $v_p(c)$ are equal.

Proof. Suppose not; without loss of generality, suppose $v_p(a) < v_p(b)$ and $v_p(a) < v_p(c)$. Then, from $a + b + c = 0$, we get

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 0,$$

or

$$a^4 + b^4 + c^4 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \implies (b^2 + c^2 - a^2)^2 = 4b^2c^2.$$

Observe that the above equation does not have square roots of integers. However, the v_p of the left-hand side is equal to $2v_p(a)$, and the v_p of the right-hand side is equal to $v_p(b) + v_p(c) > 2v_p(a)$ (since p is odd), a contradiction. ∎

Note that the lemma is still true if the hypothesis is $a + b = c$ instead of $a + b + c = 0$ by the transformation $c \to -c$.

Now, suppose $k = 4$. Let P's sides be a, b, c, and d in order, and let x and y be its diagonals. From Ptolemy's theorem, we get $ac + bd = xy$. We have $v_p(ac) \geq e$ and $v_p(bd) \geq e$, so $v_p(xy) \geq e$, or $v_p(x^2y^2) \geq 2e$. This implies that one of x^2 and y^2 is divisible by p^e; splitting P along this diagonal leads to two triangles, both of whose areas are divisible by p^e by the inductive hypothesis.

Now, consider a general k-gon $A_1A_2 \cdots A_k$, with $k \geq 5$. If the square of any of its diagonals is divisible by p^e, then we can split P along this diagonal, giving two polygons with fewer sides whose areas are divisible by p^e. Thus, we may assume that any diagonal squared has v_p less than e.

Consider a quadrilateral of the form $Q = A_iA_{i+1}A_jA_{j+1}$, where i and j are chosen such that the four vertices of Q are distinct. Let $a = A_iA_{i+1}$, $b = A_{i+1}A_j$, $c = A_jA_{j+1}$, $d = A_{j+1}A_i$, $x = A_iA_j$, and $y = A_{i+1}A_{j+1}$. By Ptolemy, we have $ac + bd = xy$. However, we have $v_p(ac) \geq e$ and $v_p(xy) < e$ (by assumption, as we assumed $v_p(x^2) < e$ and $v_p(y^2) < e$), so the lemma implies $v_p(bd) = v_p(xy)$, or

$$v_p(A_{i+1}A_j) + v_p(A_{j+1}A_i) = v_p(A_iA_j) + v_p(A_{i+1}A_{j+1}).$$

Now, sum this equation over all such quadrilaterals Q. Without loss of generality, suppose $A_1, A_2, \ldots, A_k$ are in counterclockwise order around the circle, and consider all indices modulo k. We consider what happens to each term. Note that the quadrilateral Q is valid (no two vertices are equal) if and only if $j - i \not\equiv -1, 0, 1 \pmod{k}$.

- Term $v_p(A_{i+1}A_j)$: As $j - i \not\equiv -1, 0, 1 \pmod{k}$, the terms excluded are $v_p(A_{i+1}A_{i-1})$ and $v_p(A_{i+1}A_i)$. In other words, this counts every diagonal twice except for "short" diagonals (such as $A_{i-1}A_{i+1}$ that split one vertex off from the rest) once and counts each side once.
- Term $v_p(A_{j+1}A_i)$: By symmetry between i and j, this counts everything the same number of times $v_p(A_{i+1}A_j)$ does.

- Term $v_p(A_iA_j)$: Since $j - i \not\equiv -1, 0, 1 \pmod{k}$, the terms excluded are $v_p(A_iA_{i-1})$ and $v_p(A_iA_{i+1})$. Thus, this term counts every diagonal twice and does not count any side at all.
- Term $v_p(A_{i+1}A_{j+1})$: By the transformation $i \to i-1$ and $j \to j-1$, this counts everything the same number of times $v_p(A_iA_j)$ does (note that $j - i \not\equiv -1, 0, 1 \pmod{k}$ is still preserved).

Thus, our sum equation has (after dividing by 2 to account for the symmetry present)

- on the left-hand side, each side once, each short diagonal once, and each other diagonal twice, and
- on the right-hand side, each diagonal twice.

This means that, after simplifying, our sum equation reduces to

$$\sum_{i=1}^{k} v_p(A_iA_{i+1}) = \sum_{i=1}^{k} v_p(A_iA_{i+2}).$$

However, note that every term on the left-hand side is at least $\frac{1}{2}e$. Since both sides have k terms, some term $v_p(A_rA_{r+2})$ on the right-hand side must be at least $\frac{1}{2}e$. Then, $(A_rA_{r+2})^2$ is divisible by $n = p^e$, a contradiction, and we are done. □

This problem was proposed by Russia.

4. (Solution written by Allen Liu.)

We claim the answer is $b = 6$.

When $b = 6$, $a = 196$ satisfies the desired conditions:

$$19 \mid P(197),\ P(201)$$
$$7 \mid P(198),\ P(200)$$
$$3 \mid P(199),\ P(202).$$

Now, we show we need $b \geq 6$. Note $P(n)$ is always odd. Compute the following greatest common divisors (gcd's) with the Euclidean algorithm.

$$\begin{aligned}\gcd(P(n), P(n+1)) &= \gcd(n^2+n+1, n^2+3n+3)\\ &= \gcd(2(n+1), n(n+1)+1) = 1 \qquad (1)\end{aligned}$$

$$\gcd(P(n), P(n+2)) = \gcd(n^2+n+1, n^2+5n+7) = \gcd(2(2n+3), n^2+n+1)$$

But $n^2+n+1 = (2n+3)(\frac{n}{2} - \frac{1}{4}) + \frac{7}{4}$. So we see that

$$\gcd(P(n), P(n+2)) \mid 7 \qquad (2)$$

Similarly,

$$\gcd(P(n), P(n+3)) = \gcd(n^2+n+1, n^2+7n+13) = \gcd(6(n+2), n^2+n+1)$$

Since $n^2+n+1 = (n+2)(n-1)+3$, the above expression divides $6 \times 3 = 18$ However, checking $n = 1, 2, 3, \ldots, 9$, we note that $P(n)$ is never a multiple of 9, so we get

$$\gcd(P(n), P(n+3)) \mid 3. \qquad (3)$$

Now, we finish the problem with some casework. Clearly, $b \geq 2$. If $b = 2$ or $b = 3$, the set cannot be fragrant by (1).

If $b = 4$, we see that $P(n+2)$ is relatively prime to $P(n+1)$ and $P(n+3)$, and thus, $\gcd(P(n+2), P(n+4)) = 7$. Similarly, $\gcd(P(n+1), P(n+3)) = 7$. However, $\gcd(P(n+2), P(n+3)) = 1$, so the above is impossible. Hence, the set cannot be fragrant.

Finally, we consider $b = 5$. $P(n+3)$ must have a prime factor in common with either $P(n+1)$ or $P(n+5)$ since it is relatively prime to $P(n+2)$ and $P(n+4)$. Thus, $P(n+3)$ must be a multiple of 7. Then, $P(n+2)$ and $P(n+4)$ cannot be multiples of 7 (by (1)). Thus, $\gcd(P(n+2), P(n+4)) = 1$ (by (2)). $P(n+2)$ is relatively prime to $P(n+1)$ and $P(n+3)$, so it must share a common factor with $P(n+5)$. Hence, $\gcd(P(n+2), P(n+5))$ must equal 3 (by (3)). Similarly, $\gcd(P(n+1), P(n+4))$ must also be 3. However, $P(i)$ is a multiple of 3 only when i is 1 modulo 3, so the above is impossible. Hence, the set cannot be fragrant for $b = 5$.

We have now proven that we need $b \geq 6$ and are done. ■

This problem was proposed by Luxembourg.

5. (Solution written by Junyao Peng.)

The least value of k is 2016.

First, we prove that, when $k < 2016$, the equation always has a real root. In fact, since $k \leq 2015$ and there are 2,016 different linear factors, at least one of these factors is not erased on both sides. Therefore, one real root of the equation is the root of this linear factor.

On the other hand, when $k = 2016$, we consider the following construction, which we will prove has no real roots:

$$\prod_{k=1}^{504}(x-4k+3)(x-4k) = \prod_{k=1}^{504}(x-4k+2)(x-4k+1) \tag{4}$$

Since none of $x = 1, 2, \ldots, 2{,}016$ are roots of (4), in searching for roots, we can rewrite the equation as

$$\prod_{k=1}^{504} \frac{(x-4k+3)(x-4k)}{(x-4k+2)(x-4k+1)} = 1,$$

or

$$\prod_{k=1}^{504}\left(1 - \frac{2}{(x-4k+2)(x-4k+1)}\right) = 1. \tag{5}$$

We now split into cases, depending on where a potential noninteger root x lies.

Case I: $x \notin \bigcup_{k=1}^{504}(4k-2, 4k-1)$.

Then for any $1 \leq k \leq 2016$,

$$1 - \frac{2}{(x-4k+2)(x-4k+1)} < 1.$$

In addition, if there exist distinct integers k_1 and k_2 such that for both $i \in \{1, 2\}$

$$1 - \frac{2}{(x-4k_i+2)(x-4k_i+1)} < 0$$

holds, then $(x-4k_i+2)(x-4k_i+1) < 2$, and $x \in (4k_i-3, 4k_i-2) \cup (4k_i-1, 4k_i)$.

However, since $k_1 \neq k_2$, we must have

$$((4k_1-3, 4k_1-2) \cup (4k_1-1, 4k_1)) \cap ((4k_2-3, 4k_2-2) \cup (4k_2-1, 4k_2)) = \emptyset,$$

a contradiction! So, there is at most one integer k such that

$$1 - \frac{2}{(x-4k+2)(x-4k+1)} < 0.$$

Therefore,

$$\prod_{k=1}^{504}\left(1 - \frac{2}{(x-4k+2)(x-4k+1)}\right) < 1,$$

and there are no roots to (5) in this case.

Case II: $x \in \bigcup_{k=1}^{504}(4k-2, 4k-1)$.

Suppose that $x \in (4t-2, 4t-1)$. Then

$$1 - \frac{2}{(x-4t+2)(x-4t+1)} \geq 1 + \frac{2}{((x-4t+2)-(x-4t+1))^2} = 9.$$

And for $1 \leq k \leq 2016$, $k \neq t$, it is easy to show that

$$0 < 1 - \frac{2}{(x-4k+2)(x-4k+1)} < 1.$$

Therefore, by Bernoulli's inequality,

$$\begin{aligned}
&\prod_{\substack{k=1\\k\neq t}}^{504}\left(1 - \frac{2}{(x-4k+2)(x-4k+1)}\right)\\
&\geq 1 - \sum_{\substack{k=1\\k\neq t}}^{504}\frac{2}{(x-4k+2)(x-4k+1)}\\
&\geq 1 - \sum_{k=1}^{t-1}\frac{2}{((4t-2)-4k+2)((4t-2)-4k+1)}\\
&\quad - \sum_{k=t+1}^{504}\frac{2}{((4t-1)-4k+2)((4t-1)-4k+1)}\\
&= 1 - \sum_{k=1}^{t-1}\frac{2}{(4t-4k)(4t-4k-1)} - \sum_{k=t+1}^{504}\frac{2}{(4k-4t-1)(4k-4t)}\\
&= 1 - \sum_{i=1}^{t-1}\frac{2}{(4i)(4i-1)} - \sum_{j=1}^{504-t}\frac{2}{(4j)(4j-1)}\\
&\geq 1 - 2\sum_{i=1}^{504}\frac{2}{(4i)(4i-1)}\\
&\geq 1 - 2\cdot\frac{2}{4\cdot 3} - 2\sum_{i=2}^{504}\frac{2}{(4i+1.5)(4i-2.5)}\\
&= \frac{2}{3} - \sum_{i=2}^{504}\left(\frac{1}{4i-2.5} - \frac{1}{4i+1.5}\right)
\end{aligned}$$

$$\geq \frac{2}{3} - \frac{1}{8 - 2.5}$$
$$> \frac{1}{3}.$$

So

$$\prod_{k=1}^{504} \left(1 - \frac{2}{(x - 4k + 2)(x - 4k + 1)}\right) > 9 \cdot \frac{1}{3} = 3,$$

and (5) has no solutions in this case either.

Since neither case produces any roots, we conclude that equation (5) has no real roots. Above all, we conclude that the minimum value of k is 2,016. ■.

This problem was proposed by Russia.

6. (Solution written by Yuan Yao.)

Consider a sufficiently large circle that contains all segments on its interior, and extend all n segments on both sides until they intersect the circle at $2n$ distinct points. Since before the extension each pair of segments already intersect once, the extensions will not create any more intersections, and it is easy to see that one can replace the original $2n$ endpoints with the $2n$ points on the circle and the problem will remain equivalent.

Now consider one of the chords, say PQ. Since all other chords intersect this chord on its interior, it follows that there are exactly $n - 1$ points on the interior of either of the two arcs that points P and Q separate the circle into. Because this is true for all chords, one can label the points on the circle $0, 1, 2, \dots, 2n - 1$ in clockwise order such that every two points that have the same remainder modulo n are the endpoints of the same segment.

(For the remainder of this solution, an *arc* refers to a "minor arc," or an arc that contains no more than n points on its interior or on its ends. In particular, an arc will not contain both endpoints of any chord.)

Now, we prove the following two lemmas. For both lemmas, assume that Geoff put a frog on point A and B, respectively, where the two corresponding chords are AC and BD, intersecting at point X. Notice that the two frogs will simultaneously occupy the same point (which can only be X) if and only if the number of intersection points on the interior of AX and BX are the same.

Lemma 1. If there are no other points on arc AB, then the two frogs will occupy X at the same time.

Proof. Since there are no other points on AB, there are no points on the interior of CD as well. Therefore, all other chords have one endpoint on arc BC and one on arc DA. It is not difficult to see that each chord must either intersect both segments AX and BX or intersect both segments CX and DX. (No chords other than AC and BD can go through X because no three chords meet at the same point.) This implies that the number of intersections on AX and BX must be the same, so the two frogs will occupy X at the same time. ■

Lemma 2. If there is an odd number of points on the interior of arc AB, then the two frogs will never occupy X at the same time.

Proof. Due to the way we label the endpoints on the circle, we can see easily that a chord must have its endpoints either on BC and DA or on AB and CD. Similar to the proof to Lemma 1, we can see that the chords in the first category have no effect on the difference between the number of intersections on AX and that on BX. As of the chords in the second category, it is clear that each of them must either intersect both segments AX and DX or intersect both segments BX and CX. More importantly, this means that each chord in this category creates one intersection on either AX or BX but *not both*.

Since there is an odd number of points on AB, there is an odd number of chords in the second category, which means that the number of intersections on AX and BX cannot be the same, so the two frogs will never occupy X at the same time. ■

Now, we finish the problem from here.

(a) When n is odd, let $n = 2k + 1$ for some positive integer k. Geoff can put frogs on points $0, 2, 4, \dots, 4k$, which obviously all belong to different chords. It is also not difficult to verify that there is an odd number of points on the interior of any arc whose endpoints are two of the chosen points, so by Lemma 2, no two frogs will ever occupy the same point, and Geoff is happy (a.k.a. fulfilled his wish).

(b) When n is even, let $n = 2k$ for some positive integer k. By Lemma 1, Geoff cannot choose two adjacent points on the circle as starting points, as the two frogs starting from these two points will eventually occupy the same point. On the other hand, since he needs to choose n (or $2k$) endpoints out of $2n$ (or $4k$) points total, he will be forced to choose every other point. Without loss of generality, assume that he chooses points $0, 2, 4, \dots, 4k - 2$. However, this violates the problem requirement because points 0 and $2k$ are two endpoints of the same segment (and so are points 2 and $2k + 2$, …, $2k - 2$ and $4k - 2$). Therefore, Geoff cannot fulfill his wish in this case.

Since we finished the proof for both n odd and n even, we are done. ■

This problem was proposed by Czech Republic.

Results. The International Mathematical Olympiad was held in Hong Kong on July 11–12, 2015. There were 602 competitors from 109 countries and regions. On each day, contestants were given four and a half hours for three problems.

The top score of 42/42 was achieved by six students, three from the Republic of Korea, two from the United States of America (Allen Liu and Yuan Yao), and one from the People's Republic of China.

The USA team won six gold medals, earning a total of 214 points, placing first in the world for the second time in a row. The second-place team (Republic of Korea) earned a total of 207 points, with four gold and two silver medals. This was the first back-to-back win for the USA, with previous wins in 2015, 1994, 1986, 1981, and 1977. Po-Shen Loh served as the team leader (National Coach), and Razvan Gelca served as the deputy team leader.

The students' individual results were as follow.

- Ankan Bhattacharya, who finished 11$^{\text{th}}$ grade at International Academy East in Troy, MI, won a gold medal.
- Michael Kural, who finished 12$^{\text{th}}$ grade at Greenwich High School in Riverside, CT, won a gold medal.
- Allen Liu, who finished 12$^{\text{th}}$ grade at Penfield Senior High School in Penfield, NY, won a gold medal.
- Junyao Peng, who finished 11$^{\text{th}}$ grade at Princeton International School of Mathematics and Science in Princeton, NJ, won a gold medal.
- Ashwin Sah, who finished 11$^{\text{th}}$ grade at Jesuit High School in Portland, OR, won a gold medal.
- Yuan Yao, who finished 11$^{\text{th}}$ grade at Phillips Exeter Academy in Exeter, NJ, won a gold medal.

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